Miguel Yulo

PROB SET 1

1) I am grateful to Joseph Roddas for his valuable insights.

(1) A set can have elements which are themselves sets, e.g. $\{1, \{2, 3\}, 4\}$. A set can also contain itself as one of its elements, in which case it would be called an *abnormal* set. Any set that does not contain itself as an element is called *normal*. Now consider the set N of all normal sets. Is N normal or abnormal? Show that if N is normal, then it must be abnormal. Show also that if N is abnormal, then it must be normal.

A normal set may be defined as

$$n \notin n$$

An abnormal set may be defined as

$$n \in n$$

Note that these two definitions are propositions if and only if n is a set.

Consider a set N of all normal sets.

We define N as

$$\exists N : \forall n : (n \in N \iff \underbrace{n \notin n}_{\text{Definition of normal set}})$$

So, is N a normal or abnormal set? Is N a set at all?

Let's assume N is normal, so we assume that it is true that

$$N \notin N$$

By the definition of $\notin$

$$N \notin N \text{ is true} \underset{\text{Def'n of } \notin}{\Longrightarrow} \neg(N \in N)$$

But by the definition of N

$$\neg(N \in N) \underset{\substack{\text{Def'n} \\ \text{of } N}}{\Longrightarrow} \underbrace{N \in N}_{\substack{\text{Def'n of} \\ \text{abnormal} \\ \text{set.}}}$$

This is a contradiction.
This "shows" that if N is normal, then it is abnormal

Let's now assume it is true that

$$N \in N,$$

or equivalently that N is abnormal.

By the definition of $\in$

$$N \in N \text{ is true} \underset{\substack{\text{Def'n} \\ \text{of } N}}{\Longrightarrow} \neg(N \notin N)$$

This is once again a contradiction.
This "shows" that if N is abnormal then it is normal.

From these two contradictions, we may not evaluate or decide if the proposition

$$N \in N$$

is true.

$\therefore$ We conclude that N is not a set.

2)

(2) [The difference between two sets A and B, denoted $A - B$, is the set of elements in A and not in B, i.e. $A - B := A \cap B^c$, where B^c is the complement of B. Decide if the following statement is true or false:

$$A - (B \cap C) = (A - B) \cap (A - C).$$

If it is true, prove the statement; else, provide a counterexample.

Diagram 1: $A - B := A \cap B^c$

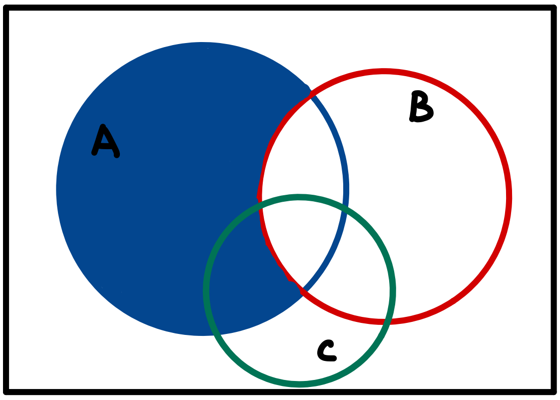


Diagram 2: $A - (B \cap C)$

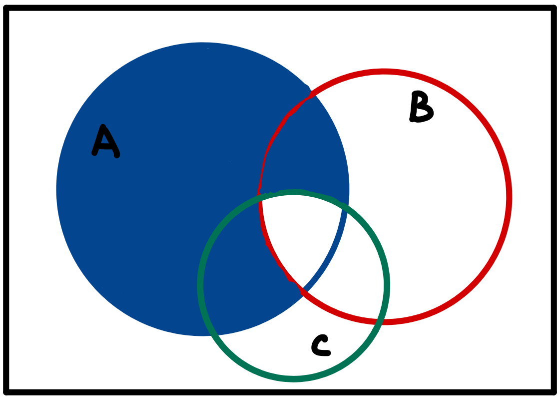


Diagram 3: $A - C$

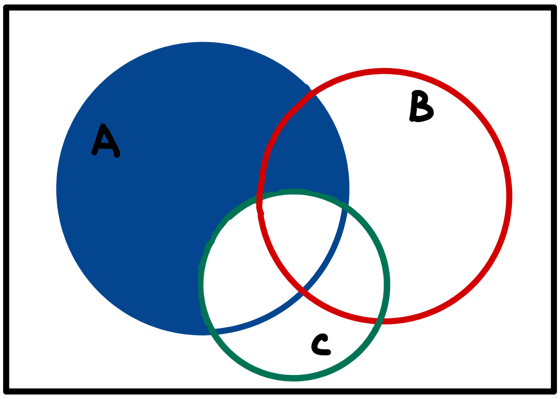
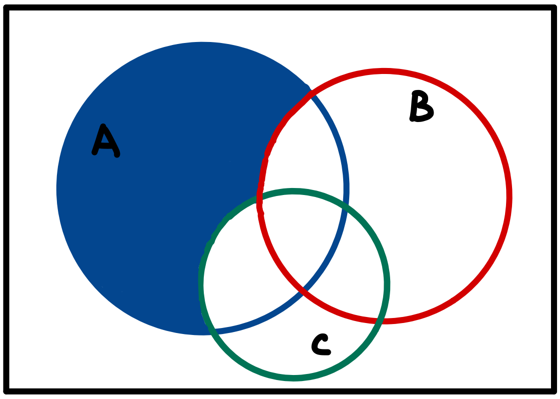


Diagram 4: $(A-B) \cap (A-C)$



We are tasked to evaluate if the following statement is true or false:

$$A-(B \cap C) = (A-B) \cap (A-C)$$

The LHS is represented by Diagram 2. The RHS is represented by Diagram 4.

As can easily be seen, Diagrams 2 and 4 are not, in general, equal. Thus, the given statement is false.

The inequality of Diagrams 2 and 4 serves as our counterexample.

If the diagrammatic illustration of a counterexample is insufficient for the reader, let us illustrate with the following:

Let

$$A = \{1,2,3,4,5\}$$
$$B = \{2,3,4\}$$
$$C = \{1,4,5\}$$

Then

$$A - B = \{1,5\}$$

$$A - C = \{2,3\}$$

$$B \cap C = \{4\}$$

$$(A-B) \cap (A-C) = \phi$$

$$A-(B \cap C) = \{1,2,3,5\}$$

$$\{1,2,3,5\} \neq \phi$$

$$\therefore A-(B \cap C) \neq (A-B) \cap (A-C).$$

3)

(3) The symmetric difference between two sets A and B is defined by $A \triangle B := (A - B) \cup (B - A)$. Decide if the following statement is true or false:

$$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$$

If it is true, prove the statement; else, provide a counterexample.

Diagram 1: $A - B$

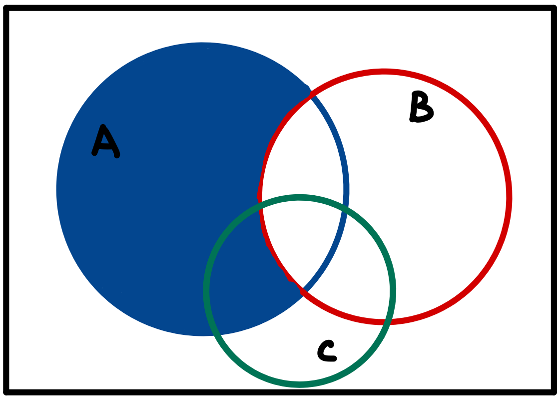


Diagram 2: $B - A$

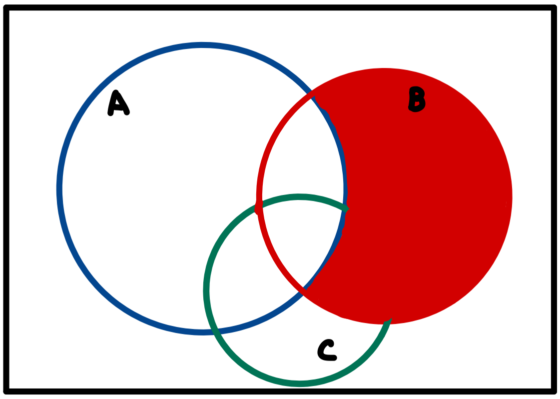


Diagram 3: $A \triangle B$

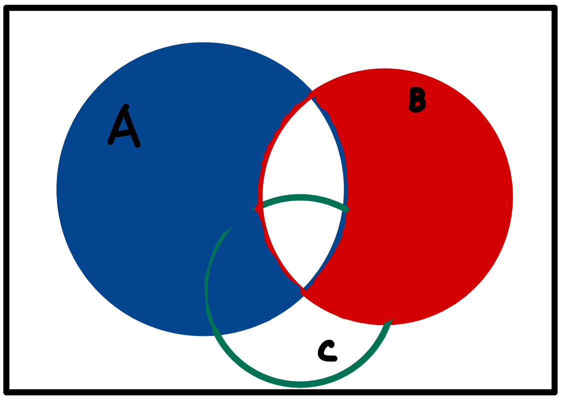


Diagram 4: $A \cap B$

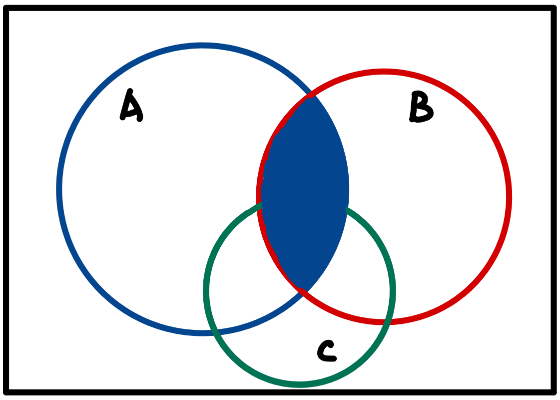


Diagram 5: $A \cap C$

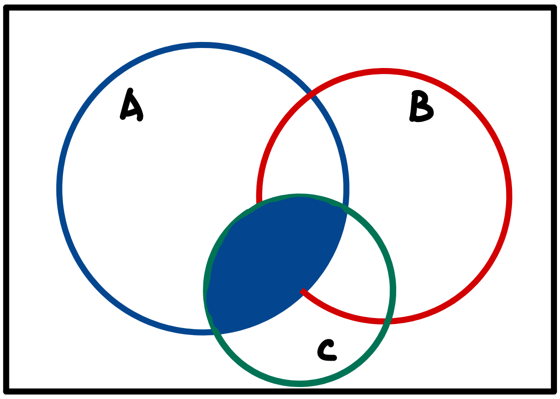


Diagram 6: $(A \cap B) - (A \cap C)$

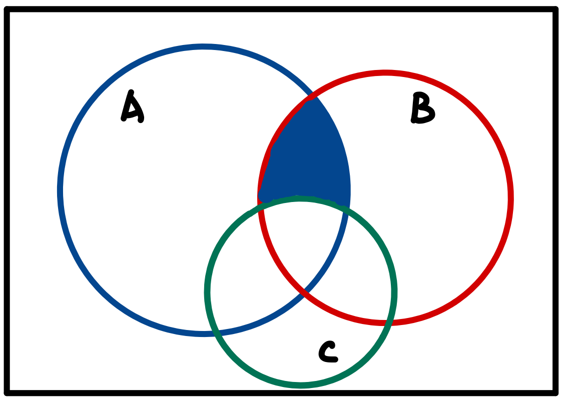


Diagram 7: $(A \cap C) - (A \cap B)$

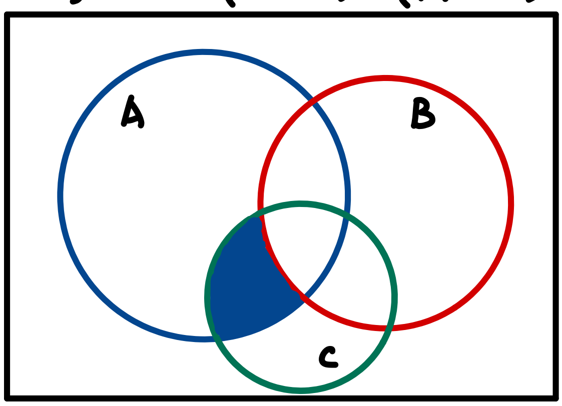


Diagram 8: $(A \cap B) \Delta (A \cap C)$

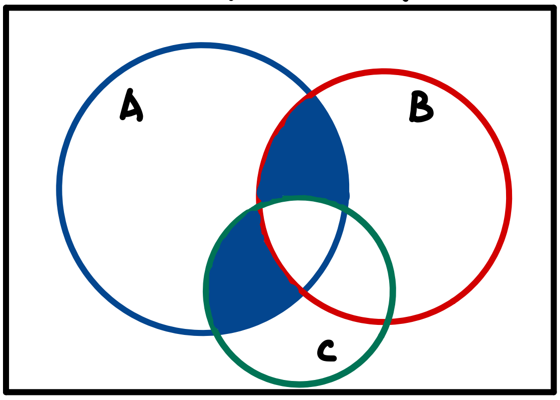


Diagram 9: $B - C$

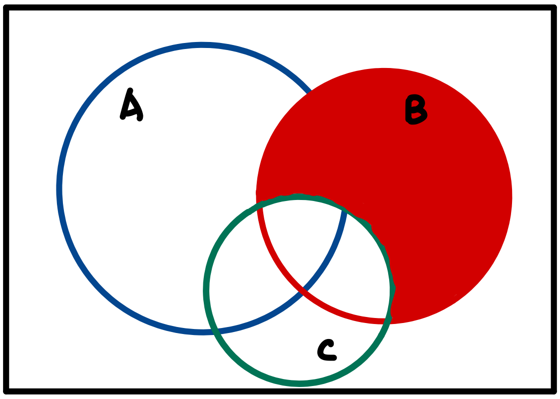


Diagram 10: $C - B$

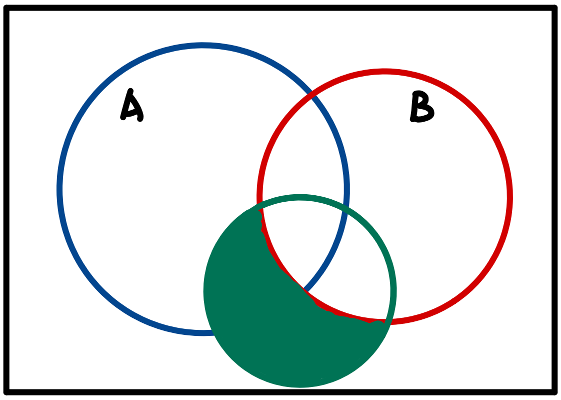


Diagram 11: $B \Delta C$

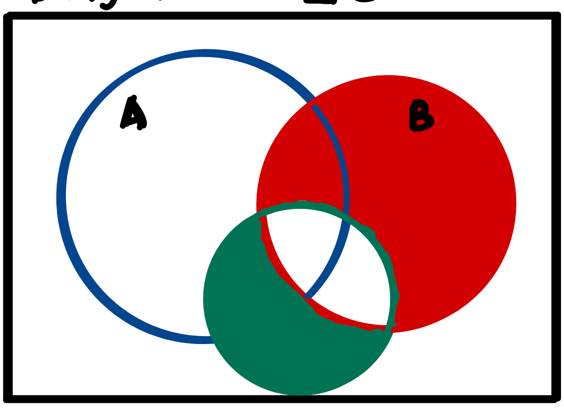
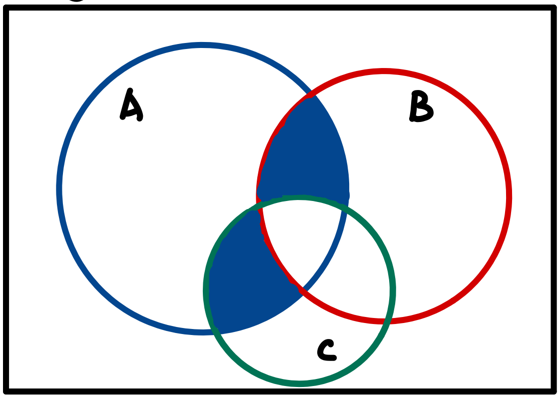


Diagram 12: $A \cap (B \Delta C)$



We are asked to prove the statement

$$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C).$$

The LHS is shown in Diagram 12.
The RHS is shown in Diagram 8.

It can be seen the two Diagrams are equal.

Thus, it has been demonstrated that

$$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$$

is true.

4)

(4) Let $f : X \to Y$ be an arbitrary mapping. Define a relation in X as follows: $x_1 \sim x_2$ means that $f(x_1) = f(x_2)$. Show that this is an equivalence relation and describe its equivalence sets.

Let X and Y be sets.
Let

$$f : X \to Y,$$

be an arbitrary mapping.

We define a relation in X

$$X_1 \sim X_2$$
means that
$$f(X_1) = f(X_2).$$

We must show that this is an equivalence relation.
We must show it is reflexive, symmetric, and transitive.

Reflexivity

By the given definition,

$$(X_1 \sim X_2) \Rightarrow f(x_1) = f(x_2)$$

Thus,

$$\forall x_1 \in X : f(x_1) = f(x_1)$$

$\therefore$ This relation is reflexive.

Symmetric

$$x_1 \sim x_2 \Rightarrow (f(x_1) = f(x_2))$$

$$f(x_1) = f(x_2) \Leftrightarrow f(x_2) = f(x_1)$$

$$f(x_2) = f(x_1) \Rightarrow (x_2 \sim x_1)$$

$$x_1 \sim x_2 \Leftrightarrow x_2 \sim x_1$$

$\therefore$ This relation is symmetric.

Transitivity

We know $x_1 \sim x_2 \Rightarrow (f(x_1) = f(x_2))$

Now let, $x_2 \sim x_3$.

Thus,

$$x_2 \sim x_3 \Rightarrow (f(x_2) = f(x_3))$$

$$x_1 \sim x_2 \wedge x_2 \sim x_3$$
$$\Rightarrow f(x_1) = f(x_2) \wedge f(x_2) = f(x_3)$$

$$\Rightarrow f(x_1) = f(x_3)$$
$$\Rightarrow x_1 \sim x_3$$

$\therefore$ This relation is transitive.

$\therefore$ Since this relation is reflective, symmetric, and transitive it is an Equivalence Relation.

Now, we must describe the equivalence classes and Quotient set.

We may define the equivalence classes

Each equivalence class contains all $x \in X$ s.t. applying the map gives the same value $f(x) = y \in Y$.

This entails that each equivalence class pertains to a distinct value of y.

$$[x_1] = \{x_1 \in X \mid f(x_1) = y_1 \in Y\}$$
$$[x_2] = \{x_1 \in X \mid f(x_1) = y_1 \in Y\}$$

$$[x_n] = \{\forall x_n \in X : \exists!\, y_n \in Y \mid f(x_n) = y_n\}$$

In the given, $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$,

$$[x_1] = [x_2]$$

The Quotient Set is thus

$$X/\sim \; = \{[x] \mid \exists!\, y \in Y : \forall x \in X, y = f(x)\}$$

5)

(5) In the set $\mathbb{R}$ of real numbers, let $x \sim y$ mean that $x - y$ is an integer. Show that this is an equivalence relation and describe the equivalence sets.

In $\mathbb{R}$, we define the relation

$$x \sim y \Rightarrow (x-y) \in \mathbb{Z}$$

We must show that this is an equivalence relation.

Reflexivity

$$\forall x \in \mathbb{R} : (x-x) \in \mathbb{Z}$$

$$x - x = 0 \in \mathbb{Z}$$

$$\Rightarrow \quad x \sim x$$

$\therefore$ This relation is reflexive.

Symmetric

$$x \sim y \Rightarrow (x-y) \in \mathbb{Z}$$

$$y \sim x \Rightarrow y - x$$

Let $x - y = b$,
then $b \in \mathbb{Z}$.

$$b \in \mathbb{Z} \Rightarrow b - (x-y) = 0 \in \mathbb{Z} \Leftrightarrow -b = y - x \in \mathbb{Z}$$

Thus,

$$\forall x, y \in \mathbb{R} : x - y \in \mathbb{Z}, \; x \sim y \Leftrightarrow y \sim x$$

$\therefore$ This relation is symmetric.

Transitivity

Let $y \sim z$

$$y \sim z \Rightarrow y - z \in \mathbb{Z}$$

Let $y - z = c \in \mathbb{Z}$,

then clearly

$$x - y + y - z = x - z = b + c \in \mathbb{Z}$$

This is ensured by the closure property of integers under addition.

Thus, $x \sim z \Rightarrow x - z \in \mathbb{Z}$.

$\therefore \forall x, y \in \mathbb{R} : x - y \in \mathbb{Z},\ x \sim y \land y \sim z \Rightarrow x \sim z$

$\therefore$ The relation is transitive.

$\therefore$ Since this relation is reflective, symmetric, and transitive it is an Equivalence Relation.

We now need to define equivalence classes and the quotient set.

Let us define a function frac(x) that gives the fractional of a real number like so:

$$\text{frac}(1) = 0, \quad \text{frac}(1.25) = 0.25, \quad \text{frac}(3.32) = 0.32$$

etc.

The range of frac is therefore $[0,1)$.

We can now see that the Equivalence Relation holds

$$\forall x, y \in \mathbb{R} : \text{frac}(x) = \text{frac}(y)$$

Let us define an a s.t.

$$\text{frac}(x) = \text{frac}(y) = a$$

Then we may define equivalence classes

$$[a] = \{ a \in [0,1) \mid x - y \in \mathbb{Z} \}$$

For example,

$$[0] = \{ \dots, -1, 0, 1, 2, \dots \} = \mathbb{Z}$$

$$[0.1] = \{ \dots, -1.1, 0.1, 1.1, 2.1, \dots \}$$

$$[0.75] = \{ \dots -2.75, -1.75, 0.75, 1.75, \dots \}$$

The Quotient Set then yields

$$\mathbb{R}/\sim \; = \{ [a] \mid a \in [0,1) \}$$

SET 2 Miguel Yulo 2018-04845

I am grateful to

Joseph Rodelas, Ronald Panganiban, and Cedric Oña for their valuable insights and answers to my questions.

I also refer to a paper from Benoumhani which I cite.

I certify that the following work is my own.

Miguel Y

1)

(1) [MC, Exercise 3.1] How many topologies can be put on: i) a set that has 2 points? ii) a set that has 3 points? iii) a set that has 4 points?

By definition, a topological space is the set X and $\mathcal{O}$, the topology on X written as a double $(X, \mathcal{O})$.

$(X, \mathcal{O})$ satisfy the following conditions:

i) $\phi \in \mathcal{O}$ where ϕ is open, and $X \in \mathcal{O}$ where X is open
ii) $A, B \in \mathcal{O}$ means $A \cap B \mathcal{O}$, so $A \cap B$ is open
iii) Let C be an arbitrary index set. Given $V_\alpha \in \mathcal{O}$, we hold that $\bigcup_{\alpha \in C} A_\alpha \in \mathcal{O}$, and is thus open.

Let $T(n)$ be the number of topologies that can be put on a set with n points/elements.

Thus, for
$\underline{n=1}$: $X = \{1\}$
We only find

$$\mathcal{O}: \{\phi, \{1\}\} = \{\phi, X\} \Rightarrow \boxed{T(1) = 1}$$

$\underline{n=2}$: $X = \{1,2\}$

$$\mathcal{O}: \{\phi, \{1,2\}\} = \{\phi, X\}$$
$$\{\phi, \{1\}, \{1,2\}\}$$
$$\{\phi, \{2\}, \{1,2\}\}$$
$$\{\phi, \{1\}, \{2\}, \{1,2\}\}$$

$$\Rightarrow \boxed{T(2) = 4}$$

$\underline{n=3}$: $X = \{1,2,3\}$

$\mathcal{O}$: $\{\emptyset, \{1,2,3\}\} = \{\emptyset, X\}$ => Trivial => 1

$\{\emptyset, \{1\}, X\}$
$\{\emptyset, \{2\}, X\}$ => Using only singletons => 3
$\{\emptyset, \{3\}, X\}$

$\{\emptyset, \{1,2\}, X\}$
$\{\emptyset, \{1,3\}, X\}$ => Using only doubles => 3
$\{\emptyset, \{2,3\}, X\}$

$\{\emptyset, \{1\}, \{1,2\}, X\}$
$\{\emptyset, \{2\}, \{1,2\}, X\}$
$\{\emptyset, \{1\}, \{1,3\}, X\}$
$\{\emptyset, \{3\}, \{1,3\}, X\}$ => Singleton and one double it is in => 6
$\{\emptyset, \{2\}, \{2,3\}, X\}$
$\{\emptyset, \{3\}, \{2,3\}, X\}$

$\{\emptyset, \{1\}, \{1,2\}, \{1,3\}, X\}$
$\{\emptyset, \{2\}, \{1,2\}, \{2,3\}, X\}$ => Singleton and two doubles it is in => 3
$\{\emptyset, \{3\}, \{1,3\}, \{2,3\}, X\}$

$\{\emptyset, \{1\}, \{2,3\}, X\}$

$\{\emptyset, \{2\}, \{1,3\}, X\}$ $\Rightarrow$ Singleton and one double if is not in $\Rightarrow 3$

$\{\emptyset, \{3\}, \{1,2\}, X\}$

$\{\emptyset, \{1\}, \{2\}, \{1,2\}, X\}$

$\{\emptyset, \{1\}, \{3\}, \{1,3\}, X\}$ $\Rightarrow$ two singletons and double they are both in $\Rightarrow 3$

$\{\emptyset, \{2\}, \{3\}, \{2,3\}, X\}$

$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, X\}$

$\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, X\}$

$\{\emptyset, \{1\}, \{3\}, \{1,3\}, \{2,3\}, X\}$

$\{\emptyset, \{1\}, \{3\}, \{1,3\}, \{1,2\}, X\}$ $\Rightarrow$ two singletons and two doubles at least one of them is in $\Rightarrow 6$

$\{\emptyset, \{2\}, \{3\}, \{2,3\}, \{1,3\}, X\}$

$\{\emptyset, \{2\}, \{3\}, \{1,2\}, \{1,3\}, X\}$

$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$ $\Rightarrow$ Discrete Topology $\Rightarrow 1$

$\therefore \boxed{T(3) = 29}$

$\underline{n=4:}$ $X = \{1,2,3,4\}$

We note that the maximum number of subsets of a topology σ on a set X of n elements is 2^n.

For example: if $n=1 \Rightarrow 2^{n=1} = 2 \Rightarrow \phi, X$
$n=2 \Rightarrow 2^{n=2} = 4 \Rightarrow \phi, \{1\}, \{2\}, X$

We use a result from (Benoumhani, 2006).

Let k be the number of open subsets in question.

k then ranges $2 \leq k \leq 2^n$.

$$T(n) = \sum_{k \geq 2} T(n,k).$$

$T(n,2) = 1 \Rightarrow$ Trivial Topology
$T(n,3) = 2^n - 2 \Rightarrow$ Trivial topology with one n-tuple

We may use Stirling numbers of the second kind for $k > 3$.

$$S(n,k) = S_{n,k} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

This gives the number of partitions of a set of n elements into k blocks.

Considering chain topologies

$$C(n,k) = (k-1)!\,S(n,k-1)$$

Thus, Benoumhani gives formulas for $T(n,k)$ $4 \le k \le 12$.

We then plug in $n=4$. We use a table of the required Stirling numbers as reference

$S_{4,1} = 1$

$S_{4,2} = 7$

$S_{4,3} = 6$

$S_{4,4} = 1$

$S(4, k>4) = 0$, so I omit them.

$T(4,2) = 1$

$T(4,3) = 2^4 - 2 = 14$

$T(4,4) = S_{4,2} + 3!S_{4,3} = 7 + 3!(6) = 43$

$T(4,5) = 3!S_{4,3} + 4!S_{4,4} = 3!(6) + 4!(1) = 60$

$T(4,6) = 3!S_{4,3} + \frac{3}{2}(4!)S_{4,4} = 3!(6) + \frac{3}{2}(4!)(1) = 72$

$T(4,7) = \frac{9}{4}(4!)S_{4,4} = \frac{9}{4}(4!)(1) = 54$

$T(4,8) = S_{4,3} + 2 \cdot 4!\,S_{4,4} = 6 + 2 \cdot 4!(1) = 54$

$T(4,9) = \frac{5}{6} \cdot 4!\,S_{4,4} = \frac{5}{6}(4!)(1) = 20$

$T(4,10) = 4!\,S_{4,4} = 4!(1) = 24$

$T(4,11) = 0 \Rightarrow$ All terms with $k>4$

$T(4,12) = \frac{1}{2} \cdot 4!\,S_{4,4} = \frac{1}{2} \cdot 4!(1) = 12$

$$\left.\begin{matrix} T(4,13) \\ T(4,14) \\ T(4,15) \end{matrix}\right\} = 0 \Rightarrow \text{All terms with } k>4$$

$T(4,16) = 1 \Rightarrow$ Discrete topology

$$\sum T(n,k) = 355, \quad \therefore \boxed{T(4) = 355}$$

References

Benoumhani, Moussa. (2006). The Number of Topologies on a Finite Set. Journal of Integer Sequences. 9.

https://cs.uwaterloo.ca/journals/JIS/VOL9/Benoumhani/benoumhani11.pdf

2)

(2) [PR, Exercise 3.2] Let X be a topological space. Show that the following properties hold.

(a) Arbitrary intersections of closed sets are closed.

(b) Finite unions of closed sets are closed.

(c) The empty set $\emptyset$ and X are both closed.

By definition, a topological space is the set X and $\mathcal{O}$, the topology on X written as a double $(X, \mathcal{O})$.

$(X, \mathcal{O})$ satisfy the following conditions:

i) $\phi \in \mathcal{O}$ where ϕ is open, and $X \in \mathcal{O}$ where X is open
ii) $A, B \in \mathcal{O}$ means $A \cap B \mathcal{O}$, so $A \cap B$ is open
iii) Let C be an arbitrary index set. Given $V_\alpha \in \mathcal{O}$, we hold that $\bigcup_{\alpha \in C} A_\alpha \in \mathcal{O}$, and is thus open.

<u>Property a) is related to condition iii)</u>

Since an arbitrary union of open sets $A \cup B$ is open wrt. $\mathcal{O}$. By definition, its complement $X - (A \cup B)$ is closed.

We can interpret $X - (A \cup B)$ as $(X - A) \cap (X - B)$ by De Morgan's Laws.

Since

$$X - (A \cup B) = (X - A) \cap (X - B),$$

$(X - A) \cap (X - B)$ is also closed.

Since X, A, and B are open by definition, $(X - A)$ and $(X - B)$ are closed.

Thus, $(X-A) \cap (X-B)$ is an arbitrary intersection of closed sets.

We have therefore shown that given a topological space, an arbitrary intersection of closed sets is closed.

Property b) is related to condition ii).

Since $A \cap B$, an open set, is an arbitrary intersection, the complement $X-(A \cap B)$ is closed.

We can interpret $X-(A \cap B)$ as a finite union $(X-A) \cup (X-B)$.

The complement of any open set wrt to the topology σ is closed.

Since
$$X-(A \cap B) = (X-A) \cup (X-B),$$

$(X-A) \cup (X-B)$ is also closed.

Since X, A, and B are open by definition, $(X-A)$ and $(X-B)$ are closed.

Thus, $(X-A) \cup (X-B)$ is a finite union of closed sets.

We have therefore shown that in a topological space, the finite union of closed sets is closed.

Property c) is related to condition i)

Since X is open wrt. σ, by definition its complement $X - X$ is closed.

$$X - X = \emptyset.$$

Since $X - X$ is closed wrt. σ, $\emptyset$ is also closed wrt. σ.

Since $\emptyset$ is open wrt. σ, by definition its complement $X - \sigma$ is closed.

$$X - \emptyset = X.$$

Since $X - \emptyset$ is closed wrt. σ, X is also closed wrt. σ.

3)

(3) Give examples of topological spaces and sets in them that

(a) are closed, but not open;

(b) are open, but not closed;

(c) are both open and closed;

(d) are neither open nor closed.

We can give examples of these using the topological space $(\mathbb{R}, \mathcal{O}_s)$, where $\mathcal{O}_s$ is the standard topology.

Additionally, for clarity, we create an arbitrary topological space $(\mathcal{M}, \mathcal{O}_\mathcal{M})$.

We define this as follows:

$$\mathcal{M} = \{a, b, c\}$$
$$\mathcal{O}_\mathcal{M} = \{\emptyset, \{a\}, \{a,b\}, \mathcal{M}\}$$

Let us find the open sets of $\mathcal{M}$

By definition, the set of all open sets in $\mathcal{M}$ is $\mathcal{O}_\mathcal{M}$.

$$\text{Open sets in } \mathcal{M} = \mathcal{O}_\mathcal{M} = \{\emptyset, \{a\}, \{a,b\}, \mathcal{M}\}$$

To find the closed sets in $\mathcal{M}$, we take the complement of the open sets.

$$\text{Closed sets in } \mathcal{M} = \{\mathcal{M} - \emptyset, \mathcal{M} - \{a\}, \mathcal{M} - \{a,b\}, \mathcal{M} - \mathcal{M}\}$$

This yields

$$\text{Closed sets in } \mathcal{M} = \{\mathcal{M}, \{b,c\}, \{c\}, \emptyset\}$$

We may now give examples using $(\mathbb{R}, \mathcal{O}_s)$ and $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$.

Let's first introduce $\alpha, \beta \in \mathbb{R}$, s.t. $\alpha < \beta$.

a) Closed but not open

For $(\mathbb{R}, \mathcal{O}_s)$: $[\alpha, \beta]$

We show this is closed by noting its complement $\mathbb{R} - [\alpha, \beta] = (-\infty, \alpha) \cup (\beta, +\infty)$ is open.

We show this is not open by noting for any $\delta > 0$, the interval $(\alpha - \delta, \beta + \delta)$ will include a number less than α and one greater than β. Obviously, these numbers would $\notin [\alpha, \beta]$.

For $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$: $\{b, c\}$ and $\{c\}$

We find these by comparing the open and closed sets. We note that $\emptyset$ and $\mathcal{M}$ are both open and closed. So, in this topological space $\{b, c\}$ and $\{c\}$ are the only sets that are closed, but not open.

b) Open, but not closed

For $(\mathbb{R}, \mathcal{O}_s)$: (α, β)

We show this by noting its complement $\mathbb{R} - (\alpha, \beta) = (-\infty, \alpha] \cup [\beta, +\infty)$ is closed.

We know this complement is closed since $(-\infty, \alpha]$ and $[\beta, +\infty)$ both include all their boundary points, and are thus closed. Obviously, the union $(-\infty, \alpha] \cup [\beta, +\infty)$ also includes all boundary points and is closed.

For $(\mathcal{M}, \mathcal{O}_\mathcal{M})$: $\{a\}$ and $\{a,b\}$

We find these by comparing the open and closed sets. We note that $\emptyset$ and $\mathcal{M}$ are both open and closed. So, in this topological space $\{a,b\}$ and $\{a\}$ are the only sets that are open, but not closed.

c) Both open and closed

For $(\mathbb{R}, \mathcal{O}_s)$: $\emptyset$, $\mathbb{R}$

By definition, in any topological space $\emptyset$ is both open and closed. Similarly, for any $(X, \mathcal{O}_X)$, X is both open and closed. So, $\mathbb{R}$ is both open and closed.

For $(\mathcal{M}, \mathcal{O}_\mathcal{M})$: $\emptyset$, $\mathcal{M}$,

By the same definitions, $\emptyset$ and $\mathcal{M}$ are both open and closed.

d) Neither open nor closed

For $(\mathbb{R}, \mathcal{O}_s)$: $(a,b]$, $\mathbb{Q}$

To check $(a,b]$ is neither open nor closed, we take its complement. $\mathbb{R} - (a,b] = (-\infty, a] \cup (b, +\infty)$

We see $b \notin (-\infty, a] \cup (b, +\infty)$, so it does not include one of its boundary values, thus it is not closed.
If we use any $\delta > 0$, the interval
$(-\infty - \delta, a + \delta)$ includes a value greater than a.
This number $\notin (-\infty, a] \cup (b, +\infty)$, so it is not open.

Since the complement of $(a,b]$ is neither open nor closed, $(a,b]$ is also neither open nor closed.

$\mathbb{Q}$ is the set of rational numbers.

$$\mathbb{Q} \subseteq \mathbb{R}$$

For any $x \in \mathbb{Q}$ and $\delta > 0$, the interval

$(x-\delta, x+\delta)$ will include values $\notin \mathbb{Q}$.

So, $\mathbb{Q}$ is not open.

Its complement is $\mathbb{R} - \mathbb{Q}$.

For any $y \in (\mathbb{R} - \mathbb{Q})$ and $\delta > 0$, the interval

$(y-\delta, y+\delta)$ will include values $\in \mathbb{Q}$.

Thus, the complement is not open.

So, $\mathbb{Q}$ is not closed.

$\therefore$ $\mathbb{Q}$ is neither open nor closed.

<u>For $(M, \mathcal{O}_M)$: $\{b\}$</u>

$\{b\}$ is not in the open sets in M nor the closed sets in M.

$\therefore$ $\{b\}$ is neither open nor closed.

4)

(4) [MN, Exercise 2.14] By taking a continuous function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ as an example, show that the reverse definition, " a map f is continuous if it maps an open set in X to an open set in Y", does not work. [Hint: Find where $(-\varepsilon, +\varepsilon)$ is mapped to under f.]

By the given, we know the map f is continuous.

We must then just find a counter-example of the reverse definition to show that it does not work.

Let $Y \subset \mathbb{R}$ be an open set.
Let $X \subset \mathbb{R}$ be the set $f^{-1}(Y)$.

Since f is continuous, we know that $f^{-1}(Y)$ is open for any $x \in f^{-1}(Y)$.
Thus, X is open.

Since Y is open, we can find $\epsilon > 0 \in \mathbb{R}$ s.t.

$$(f(x) - \epsilon, f(x) + \epsilon) \subset Y$$

so

$$(x^2 - \epsilon, x^2 + \epsilon) \subset Y$$

for any $x \in f^{-1}(Y)$.

So take $x = 0$, $x' \in (-\epsilon, \epsilon)$

$$(-\epsilon, \epsilon) \subset X.$$

This is clearly an open set.

If we use the map f

$f : (-\epsilon, \epsilon) \mapsto [0, \epsilon^2)$, since $\epsilon > 0$.

We can show that

$[0, \epsilon^2)$ is not open since for any $\delta > 0$

the open interval $(0 - \delta, \epsilon^2 + \delta)$ will include a number less than 0. This number would obviously $\notin [0, \epsilon^2)$.

Thus, we since we know f is continuous and that it has mapped an open set in X to a non-open set in Y, then the given reverse definition does not hold.

5)

(5) [PR, Exercise 3.13] If X is a topological space (with topology τ) and $\sim$ is an equivalence relation on X, the quotient $Y := X/\sim$ is naturally a topological space under the quotient topology σ, defined as

$$\sigma = \{U \subseteq Y : \pi^{-1}(U) \in \tau\}$$

where $\pi : X \to Y$ is the natural projection map $x \mapsto [x]$. In other words, the quotient topology is the finest topology for which π is continuous. Suppose that $g : X \to Z$ is a continuous map of topological spaces that respects the equivalence relation $\sim$, so that $x_1 \sim x_2$ implies $g(x_1) = g(x_2)$. Show there is a unique continuous map $f : Y \to Z$ such that $g = f \circ \pi$. (One says that g "descends to the quotient".)

We assume (X, τ) is a topological space.

The Quotient Y is defined as

$$Y := X/\sim \text{ s.t. } (Y, \sigma) \text{ is a topological space.} \tag{1}$$

$$\sigma := \{U \subseteq Y : \pi^{-1}(U) \in \tau\} \tag{2}$$

where

$$\left.\begin{array}{l} \pi : X \to Y \quad (3) \\ \pi : x \mapsto [x] \quad (4) \end{array}\right\} \text{Continuous}$$

Suppose

$$g : X \to Z \quad \left\{\begin{array}{l} \bullet \text{ continuous} \\ \bullet \text{ respects } x_1 \sim x_2 \Rightarrow g(x_1) = g(x_2) \end{array}\right. \tag{5}$$

This implies a topology on Z that we may call σ_Z.

We are tasked to show that there exist a unique continuous map f s.t.

$$f : Y \to Z \tag{6}$$

$$f : [x] \mapsto f([x]) \tag{7}$$

$$\Updownarrow$$

$$g = f \circ \pi \tag{8}$$

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[rr, bend right, "g = f\circ\pi"'] & Y \arrow[r, "f"] & Z \end{tikzcd}$$

We may evaluate this question using the continuity of the composition of continuous maps.

Let us take a $V \in Z$, then $V \in \mathcal{O}_Z$.

To show f is continuous, we must show

$$\text{preim}_{f\circ\pi}(V) = \text{preim}_g(V) \tag{9}$$

$$\text{preim}_{f\circ\pi}(V) := \{x \in X \mid (f\circ\pi)(x) \in V\} \tag{10}$$

True if f is continuous $\Rightarrow$ $= \{x \in X \mid \pi(x) \in \text{preim}_f(V) \in \sigma\}$ (11)

True if π is continuous, $\Rightarrow$ $= \text{preim}_\pi(\text{preim}_f(V)) \in \tau$ (12)

which we know it is.

Thus, $\text{preim}_f(V) \in U$, $U \subseteq Y$ so (13)

$$\text{preim}_f(V) = \pi^{-1}(U) \in \tau. \tag{14}$$

So far (11) has yet to be shown.

By definition in (5), g respects $\sim$.

Thus if $g = f \circ \pi$

$$g : X \to Z$$

implies $g : x \to f([x])$. (15)

Since we know g is continuous, and $g = f \circ \pi$, then f must be continuous.

We already know π is continuous.

So

$$\text{preim}_g(V) := \{x \in X \mid g(x) \in V \mid x_1 \sim x_2 \Rightarrow g(x_1) = g(x_2)\} \quad (16)$$

$$= \{x \in X \mid (f \circ \pi)(x) \in V\} \quad (17)$$

$$\therefore \quad \text{preim}_g(V) = \text{preim}_{f \circ \pi}(V) \quad (18)$$

This tells us that for the condition that g respects $\sim$ to be true, g must be able to be expressed as a composition involving a map after π.

This is because π is the map establishing the said equivalence relation.
We call this new map f.
Thus, since $g = f \circ \pi$ and g is continuous, f is a unique continuous map as in (6) and (7). ■

6) (6) [MC, Exercise 3.8] Let $f : \mathbb{R} \to \mathbb{Z}$ be the "floor" function which rounds a real number x down to the nearest integer:

$$f(x) = n \text{ provided that } n \in \mathbb{Z} \text{ and } n \leq x < n+1$$

Determine whether or not f is continuous.

$$f: \mathbb{R} \to \mathbb{Z}$$

We automatically assume $\mathbb{R}$ carries the standard topology. We note that $\mathbb{Z} \subset \mathbb{R}$. We let $\mathbb{Z}$ inherit the subset topology.

Let $X \subset \mathbb{R}$ and $N \subset \mathbb{Z}$.

We define the pre-image of N under f as

$$f^{-1}(N) = \{x \in X : f(x) \in N\}.$$

For f to be continuous, whenever N is open, $f^{-1}(N)$ must also be open.

We may then show f is not continuous by finding at least one open subset of N wherein the pre-image under f is not open.

The defined f is

$$f(x) = n, \quad n \leq x < n+1, \text{ provided } n \in \mathbb{Z}.$$

Let us take some n, say $n = 1 \in N$.

We note that the set $\{1\}$ is open, due to $\mathbb{Z}$'s inherited subset topology.

We take the pre-image under f of $\{1\}$.

$$f^{-1}\{1\} = [1,2).$$

We can show $[1,2)$ is not open.

Take any $\delta > 0$.

The open interval $(1-\delta, 2+\delta)$ will include a number less than 1.
This number would evidently $\notin [1,2)$.

$\therefore$ $[1,2)$ is not open.

This may be generalized to say that $[n, n+1)$ is not open.

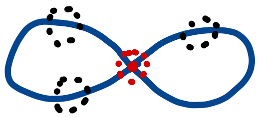
Since we have found a pre-image of an open set in $\mathbb{N}$ under f that is not open, f is not continuous.

$\blacksquare$

7)

(7) Give examples of sets that are not manifolds, and explain why they are not.

EXAMPLE 1: Consider the following set

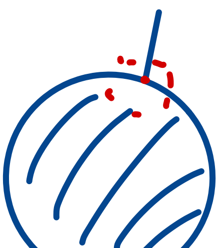
 $X \subseteq (\mathbb{R}^2, \mathcal{O}_{st})$

For a set to be a manifold in $\mathbb{R}$, it must look like $\mathbb{R}$ at all points.

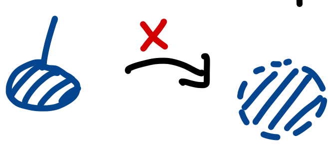
Meaning, for a 1-D manifold we should be able to take a soft ball around any point s.t. the local neighborhood looks like a line segment. However, when we take any soft ball around the red intersection point, we get 4 distinct branches. Thus, at this point it does not locally resemble a line segment. We have no problem for other points.

This also would not be a manifold in 2-D, as we cannot find an open ball about that point where it locally looks like a plane nor a line segment.

EXAMPLE 2: Consider a sphere with a line protruding

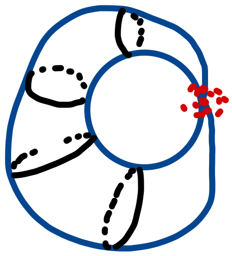
 $\subseteq \mathbb{R}^3$

If we take an open ball around the intersection point, we see that



It cannot be mapped to an open disk.

EXAMPLE 3: Consider the surface

 $\subseteq \mathbb{R}^3$

If we take an open ball (spherical) around this point intersection, it cannot be mapped to an open disk.

Thus, this is not a 2-D manifold.