Miguel Yulo 2018-04845

Problem Set 1 **Physics 225**

I would like to thank Ronald Panganiban and Gedrich Dy for some of their insights into our discussion regarding this problem set. I however certify that the solutions below are my own work.

Problem 1

Solution

We consider that the coordinate components of a vector must transform

$$x^\alpha \to y^\alpha \left(x^\beta\right) \tag{1}$$

according to

$$V'^\mu = \frac{\partial y^\mu}{\partial x^v} V^\nu . \tag{2}$$

We must show that the action of the coordinate basis one-form dx^μ on V

$$dx^\mu(V) = V^\mu \tag{3}$$

also transforms according to the formula in (2). We note that (3) is equivalent to the action of the vector V on the one-form x^μ.

$$V\left(x^\mu\right) = \frac{\partial}{\partial x^\alpha}\left(x^\mu\right) V^\alpha \tag{4}$$

$$V^\alpha \frac{\partial}{\partial x^\alpha}\left(x^\mu\right) = \delta^\mu_\alpha V^\alpha \tag{5}$$

$$V^\alpha \delta^\mu_d = V^\mu \tag{6}$$

Thus, it's clear that

$$dx^\mu(V) = V\left(x^\mu\right). \tag{7}$$

So if we show that $V\left(x^\mu\right)$ transforms according to (2), then we would have equivalently shown that $dx^\mu(V)$ transforms as such.

We find that we can enact the transformation on (4).

$$\frac{\partial}{\partial x^\alpha}\left(y^\mu\left(x^\gamma\right)\right) V^\alpha = \frac{\partial y^\mu}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^\alpha} V^\alpha \tag{8}$$

$$\frac{\partial y^\mu}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x^\alpha} V^\alpha = \frac{\partial y^\mu}{\partial x^\gamma}\delta^\gamma_\alpha V^\alpha \tag{9}$$

$$\boxed{\frac{\partial y^\mu}{\partial x^\gamma}\delta^\gamma_\alpha V^\alpha = \frac{\partial y^\mu}{\partial x^\gamma} V^\gamma} \tag{10}$$

Thus, we can see in (10) that (4) transforms like (2). Therefore, (3) also transforms like (2).

We now consider the coordinate transformation given by

$$T = \frac{c}{g}\sinh\left(\frac{gt}{c}\right) + \frac{x}{c}\sinh\left(\frac{gt}{c}\right) \tag{11}$$

$$X = \frac{c^2}{g}\left(\cosh\left(\frac{gt}{c}\right) - 1\right) + x\cosh\left(\frac{gt}{c}\right). \tag{12}$$

We seek V^T and V^X in terms of V^x and V^t.

Taking the differentials of T and X, we find

$$\underline{dT} = \cosh\left(\frac{gt}{c}\right)\underline{dt} + \frac{1}{c}\sinh\left(\frac{gt}{c}\right)\underline{dx} + \frac{xg}{c^2}\cosh\left(\frac{dt}{c}\right)\underline{dt} \tag{13}$$

$$\underline{dT} = \left(\frac{xg}{c^2}+1\right)\cosh\left(\frac{gt}{c}\right)\underline{dt} + \frac{1}{c}\sinh\left(\frac{gt}{c}\right)\underline{dx} \tag{14}$$

$$\underline{dX} = c\sinh\left(\frac{gt}{c}\right)\underline{dt} + \cosh\left(\frac{gt}{c}\right)\underline{dx} + \frac{xg}{c}\sinh\left(\frac{gt}{c}\right)\underline{dt} \tag{15}$$

$$\underline{dX} = \left(c+\frac{xg}{c}\right)\sinh\left(\frac{gt}{c}\right)\underline{dt} + \cosh\left(\frac{gt}{c}\right)\underline{dx}. \tag{16}$$

We can then use (3) on (14) and (16) to get

$$\underline{dT}(V) = \left(\frac{xg}{c^2}+1\right)\cosh\left(\frac{gt}{c}\right)\underline{dt}(V) + \frac{1}{c}\sinh\left(\frac{gt}{c}\right)\underline{dx}(V) \tag{17}$$

$$\boxed{V^T = \left[\left(\frac{xg}{c^2}+1\right)\cosh\left(\frac{gt}{c}\right)\right]V^t + \left[\frac{1}{c}\sinh\left(\frac{gt}{c}\right)\right]V^x}, \tag{18}$$

and

$$\underline{dX}(V) = \left(c+\frac{xg}{c}\right)\sinh\left(\frac{gt}{c}\right)\underline{dt}(V) + \cosh\left(\frac{gt}{c}\right)\underline{dx}(V) \tag{19}$$

$$\boxed{V^X = \left[\left(c+\frac{xg}{c}\right)\sinh\left(\frac{gt}{c}\right)\right]V^t + \left[\cosh\left(\frac{gt}{c}\right)\right]V^x}. \tag{20}$$

Q.E.D.

Problem 2

Solution

We have the inertial coordinates

$$x^0 = t \tag{1}$$
$$x^1 = x \tag{2}$$
$$x^2 = y \tag{3}$$
$$x^1 = z. \tag{4}$$

Our Minkowski metric has the components

$$g_{\mu\nu} = diag(-1, 1, 1, 1). \tag{5}$$

We now consider the given Rindler Coordinates

$$\bar{x}^0 = \tau \tag{6}$$
$$\bar{x}^1 = \phi \tag{7}$$
$$\bar{x}^2 = y \tag{8}$$
$$\bar{x}^3 = z. \tag{9}$$

We have the transformation given by

$$t = \tau \cosh \phi \tag{10}$$
$$x = \tau \sinh \phi. \tag{11}$$

We can then use this to find the components of the metric in this coordinate system. We first start by trying to find the line element ds^2.

We get the differentials of (10) and (11).

$$dt = \cosh \phi d\tau + \tau \sinh \phi d\phi \tag{12}$$
$$dx = \sinh \phi d\tau + \tau \cosh \phi d\phi \tag{13}$$

Thus, we can "square" these to get

$$dt^2 = \cosh^2 \phi d\tau^2 + 2\tau \sinh \phi \cosh \phi d\phi d\tau + \tau^2 \sinh^2 \phi d\phi^2 \tag{14}$$
$$dx = \sinh \phi d\tau + \tau \cosh \phi d\phi. \tag{15}$$

Due to (3) being the same as (8) and (4) being the same as (9), we get

$$dy^2 = dy^2 \tag{16}$$
$$dz^2 = dz^2. \tag{17}$$

Now from (5), we know the line element ds^2 is of the form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{18}$$

Using (14), (15), (16), and (17), we arrive at

$$ds^2 = \left(\sinh^2 \phi - \cosh^2 \phi\right) d\tau^2 + \tau^2 \left(\cosh^2 \phi - \sinh^2 \phi\right) d\phi^2 + dy^2 + dz^2. \tag{19}$$

This simplifies to

$$ds^2 = -d\tau^2 + \tau^2 d\phi^2 + dy^2 + dz^2. \tag{20}$$

We can then write the components of our metric as

$$\boxed{g_{\mu\nu} = \mathrm{diag}\left(-1, \tau^2, 1, 1\right).} \tag{21}$$

We now seek to find the region of Minkowski spacetime covered by these coordinates. Since the transformation in (10) and (11) is defined by hyperbolic functions, it motivates us to use the identity

$$\cosh^2 \phi - \sinh^2 \phi = 1. \tag{22}$$

This is then applied to

$$t^2 - x^2 = \tau^2 \left(\cosh^2 \phi - \sinh^2 \phi\right) \tag{23}$$

$$t^2 - x^2 = \tau^2. \tag{24}$$

The form of (24) looks familiar, as a similar form is used to define the invariant spacetime interval in Taylor and Wheeler's Spacetime Physics textbook.

From (3), (4), (8), and (9), we see that we have no restrictions on y and z for the Rindler coordinates. We also note that (24) is the equation of a hyperbola for constant τ^2.

Noting that

$$\tau^2 \geq 0, \tag{25}$$

so that

$$x^2 = t^2 - \tau^2. \tag{26}$$

Combining (25) and (26), we get the condition

$$x^2 \leqslant t^2. \tag{27}$$

We now use this to graph the our region onto the $xt - plane$ of Minkowski spacetime on Desmos.

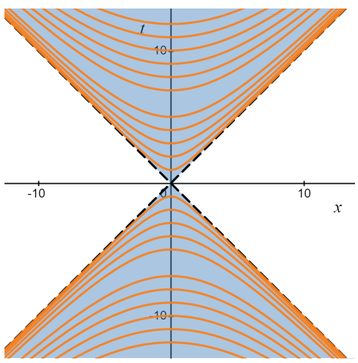


Figure 1: Region of $xt - plane$ of Minkowski spacetime covered by the Rindler Coordinates.

We can see that the orange curves are hyperbolas that plot constant values of τ^2 as we predicted. The blue regions show the possible region. Thus, the Rindler coordinates cover the region of Minkowski spacetime defined by the condition in (27), with no restrictions on y and z.
Q.E.D.

Problem 3

Solution

We have a coordinate system with metric components given by

$$g_{xx} = 1 \tag{1}$$

$$g_{yy} = \frac{1}{\cosh^4(y)} \tag{2}$$

$$g_{xy} = 0. \tag{3}$$

In matrix form this is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\cosh^4(y)} \end{pmatrix}. \tag{4}$$

We seek the geodesic equation for this coordinate system and to solve it for x and y.

We consider x and y as functions of the arbitrary parameter λ.

$$x = x(\lambda) \tag{5}$$

$$y = y(\lambda) \tag{6}$$

We denote the derivatives with respect to λ as

$$\frac{dx}{d\lambda} = \dot{x} \tag{7}$$

$$\frac{dy}{d\lambda} = \dot{y}. \tag{8}$$

We consider the Lagrangian formulation. We first get the line element from (4).

$$ds^2 = dx^2 + \frac{1}{\cosh^4(y)} dy^2 \tag{9}$$

We seek to minimize the distance on a curve given by

$$S = \int_A^B ds \tag{10}$$

$$\int_A^B ds = \int_A^B \sqrt{dx^2 + \frac{1}{\cosh^4(y)} dy^2}. \tag{11}$$

Using the parametrization, we get

$$\int_A^B \sqrt{dx^2 + \frac{1}{\cosh^4(y)} dy^2} = \int_A^B \sqrt{\dot{x}^2 + \frac{1}{\cosh^4(y)} \dot{y}^2} d\lambda. \tag{12}$$

Since the square root is monotonic, to extremize (12) we consider

$$L = \dot{x}^2 + \frac{1}{\cosh^4(y)} \dot{y}^2. \tag{13}$$

This then yields two Euler-Lagrange equations.

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial \dot{x}}\right] = \frac{\partial L}{\partial x} \tag{14}$$

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial \dot{y}}\right] = \frac{\partial L}{\partial y} \tag{15}$$

We first deal with (14).

$$\frac{\partial L}{\partial x} = 0 \tag{16}$$

$$\frac{\partial L}{\partial \dot{x}} = 2\dot{x} \tag{17}$$

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial \dot{x}}\right] = 2\ddot{x} \tag{18}$$

$$2\ddot{x} = 0 \tag{19}$$

This yields the geodesic equation in x

$$\ddot{x} = 0. \tag{20}$$

We now deal with (15).

$$\frac{\partial L}{\partial y} = \frac{-4}{\cosh^5(y)}(\sinh(y))\dot{y}^2 \tag{21}$$

$$\frac{\partial L}{\partial y} = \frac{-4}{\cosh^4(y)}\tanh(y)\dot{y}^2 \tag{22}$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{2}{\cosh^4(y)}\dot{y} \tag{23}$$

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial \dot{y}}\right] = \frac{2}{\cosh^4(y)}\ddot{y} + \frac{-8}{\cosh^4(y)}\tanh(y)\dot{y}^2 \tag{24}$$

$$\frac{2}{\cosh^4(y)}\ddot{y} + \frac{-8}{\cosh^4(y)}\tanh(y)\dot{y}^2 = \frac{-4}{\cosh^4(y)}\tanh(y)\dot{y}^2 \tag{25}$$

Solving for $\ddot{y}$, we get the geodesic equation in y

$$\ddot{y} = 2\tanh(y)\dot{y}^2. \tag{26}$$

We now seek to solve the geodesic equations (20) and (26).
We start with (20). We perform a series of integrations. Note that we will have multiple constants C_n from integration, that we will relabel from time to time.

$$\int \ddot{x}\, d\lambda = \dot{x} \tag{27}$$

$$\dot{x} = C_1 \tag{28}$$

$$\int \dot{x}\, d\lambda = x \tag{29}$$

This yields our solution for x.

$$\boxed{x(\lambda) = C_1\lambda + C_2} \tag{30}$$

We now work on (26).

We consider the identity

$$\ddot{y} = \dot{y}\frac{d\dot{y}}{dy}. \tag{31}$$

Equation (26) thus becomes

$$\dot{y}\frac{d\dot{y}}{dy} = 2\tanh(y)\dot{y}^2. \tag{32}$$

Multiplying both sides by dy and integrating, we get

$$\int \frac{dy}{\dot{y}} = \int 2\tanh(y)dy. \tag{33}$$

$$\ln|\dot{y}| = 2\ln|\cosh(y)| + C_3 \tag{34}$$

Taking the exponential of both sides of both sides of (34) and rearranging, we get

$$\dot{y} = C_4 \cosh^2(y). \tag{35}$$

We rearrange and integrate once more.

$$\frac{1}{C_4}\tanh(y) + C_5 = \lambda \tag{36}$$

$$\tanh(y) = C_4\lambda - C_4C_5 \tag{37}$$

Relabelling arbitrary constants, we get

$$\tanh(y) = C_3\lambda + C_4. \tag{38}$$

We then solve this for y to get the solution to that geodesic equation.

$$\boxed{y(\lambda) = \tanh^{-1}(C_3\lambda + C_4).} \tag{39}$$

We now introduce a coordinate transformation with a flat metric described by

$$g = dx'^2 + dy'^2. \tag{40}$$

Equating this to the original metric yields

$$dx^2 + \frac{1}{\cosh^4(y)}dy^2 = dx'^2 + dy'^2. \tag{41}$$

Looking at (30) and (40), we can assume both x and x' are linear in λ. So this motivates the putative transformation given by

$$x = x'. \tag{42}$$

Thus, (30) can be rewritten as

$$\boxed{x'(\lambda) = C_1\lambda + C_2.} \tag{43}$$

To make (38) linear, we need to free lambda from the inside of $\tanh^{-1}$. This motivates the transformation given by

$$\boxed{y' = \tanh(y).} \tag{44}$$

Applying this to (38), we get

$$y'(\lambda) = C_3\lambda + C_4. \tag{45}$$

We can easily see that (42) and (44) are equations of straight lines in λ.

We can check that (41) and (43) are indeed the correct transformations. We take their differentials.

$$dx = dx' \tag{46}$$

$$dy = \frac{1}{\cosh^2(y)}dy' \tag{47}$$

Squaring these, we do indeed get the equality of the metrics in (40).
Q.E.D.

I would like to acknowledge the help provided by Gedrich Dy and Luke Gurrea in my answering of this problem set.

I however certify that the following work is from my own efforts

Miguel Yulo

SET 2

1) (1) Let $V = V^\alpha \partial_\alpha$ be a vector field on M. Let $g = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ be a metric on M. (a) Show by transformation properties that $V_\beta := g_{\beta\alpha} V^\alpha$ are components of a covector. (b) Is $\partial_\mu V^\nu$ a tensor? (c) Does $\partial_\mu V^\nu - \partial_\nu V^\mu$ represent components of a tensor? What about $\partial_\mu V_\nu - \partial_\nu V_\mu$?

We have the vector field V on M of the form

$$V = V^\mu \partial_\mu . \tag{1}$$

We let

$$g = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

be a metric on M.

A) We want to show that something of the form

$$V_\beta := g_{\beta\alpha} V^\alpha \tag{3}$$

are components of a covector, by transformation properties.

We can take the metric from the chart

$$\{x^\mu\} \longrightarrow \{y^\beta\} . \tag{4}$$

Thus, (2) may be equivalently expressed via transformation as

$$g = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} dy^\beta dy^\alpha . \tag{5}$$

With

$$\overline{g_{\beta\alpha}} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} , \tag{6}$$

as the components in this new chart.

Similarly, we consider the transformation of the vector in (1) via the transformation in (4).

We get

$$V = \frac{\partial y^\beta}{\partial x^\mu} V^\mu \partial_\beta . \tag{7}$$

This then has components

$$\bar{V}^\beta = \frac{\partial y^\beta}{\partial x^\mu} V^\mu . \tag{8}$$

We now "act" g on V in this new chart.

Since

$$g \in (0,2) \tag{9}$$

and

$$V \in (1,0), \tag{10}$$

we expect

$$g(V, \cdot) \in (0,1) \tag{11}$$

which is a covector.

In $\{y^\beta\}$ then, the components of (11) would be

$$\bar{g}_{\alpha\beta} \bar{V}^\beta = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial x^\nu}{\partial y^\alpha} \frac{\partial y^\beta}{\partial x^\mu} V^\mu . \tag{12}$$

Using "cancellation" to simplify this, we have

$$\boxed{\bar{g}_{\alpha\beta} \bar{V}^\beta = \frac{\partial x^\nu}{\partial y^\alpha} g_{\mu\nu} V^\mu .} \tag{13}$$

We know a covector's components transform as

$$\overline{\omega}_\beta = \frac{\partial x^\mu}{\partial y^\beta} \omega_\mu \tag{14}$$

This matches the form of (13), and thus have shown the validity of (3). This also matches our prediction in (11).

B) We want to know if $\partial_\mu V^\nu$ is a tensor.
We know this is equivalent to

$$\partial_\mu V^\nu = \frac{\partial}{\partial x^\mu} V^\nu . \tag{15}$$

To find out whether or not this is like a tensor, we must find out whether it transforms like a tensor.

V^ν looks like the component of the vector, so we transform it between the charts

$$\{x^\nu\} \to \{y^\beta\} . \tag{16}$$

So like (8), we get

$$\overline{V}^\beta = \frac{\partial y^\beta}{\partial x^\nu} V^\nu . \tag{17}$$

Now ∂_μ looks like a basis vector, so it should transform covariantly, as its index is covariant (co goes below).

$$\{x^\mu\} \to \{y^\alpha\} , \tag{18}$$

as

$$\overline{\partial}_\alpha = \frac{\partial}{\partial y^\alpha} \tag{19}$$

So, "like" (14),

$$\bar{\partial}_\alpha = \frac{\partial x^\mu}{\partial y^\alpha}\partial_\mu \qquad (20)$$

$$\bar{\partial}_\alpha = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial}{\partial x^\mu}. \qquad (21)$$

So now we want to see if $\bar{\partial}_\alpha \bar{V}^\beta$ looks like a tensor

$$\bar{\partial}_\alpha \bar{V}^\beta = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial}{\partial x^\mu}\left(\frac{\partial y^\beta}{\partial x^\nu}V^\nu\right) \qquad (22)$$

By the product rule

$$\bar{\partial}_\alpha \bar{V}^\beta = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\nu}\frac{\partial}{\partial x^\mu}(V^\nu) + V^\nu\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial}{\partial x^\mu}\left(\frac{\partial y^\beta}{\partial x^\nu}\right). \qquad (23)$$

$$\bar{\partial}_\alpha \bar{V}^\beta = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^\nu}\partial_\mu V^\nu + \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial^2 y^\beta}{\partial x^\mu \partial x^\nu}V^\nu \qquad (24)$$

The first term on the RHS of (24) looks like the transformation of a (1,1) tensor. However, the additional second term screws this up. So unless the second term is zero, $\boxed{\partial_\mu V^\nu \text{ is not a tensor/}}$ nor like components of a tensor.

The second derivative is not how we expect tensors to transform.

c) We now want to check if

$$\partial_\mu V^\nu - \partial_\nu V^\mu = \frac{\partial}{\partial x^\mu} V^\nu - \frac{\partial}{\partial x^\nu} V^\mu \tag{25}$$

are the components of a tensor.

The transformation of $\partial_\mu V^\nu$ is given by (24).

∂_ν transforms as

$$\overline{\partial_\beta} = \frac{\partial x^\nu}{\partial y^\beta} \partial_\nu \tag{26}$$

$$\overline{\partial_\beta} = \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial}{\partial x^\nu} . \tag{27}$$

V^μ transforms as

$$\overline{V^\alpha} = \frac{\partial y^\alpha}{\partial x^\mu} V^\mu \tag{28}$$

So $\partial_\nu V^\mu$ transforms as

$$\overline{\partial_\beta} \overline{V^\alpha} = \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial}{\partial x^\nu} \left(\frac{\partial y^\alpha}{\partial x^\mu} V^\mu \right) \tag{29}$$

By product rule

$$\overline{\partial_\beta} \overline{V^\alpha} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial}{\partial x^\nu} (V^\mu) + V^\mu \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\mu} \tag{30}$$

$$\overline{\partial_\beta} \overline{V^\alpha} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^\beta} \partial_\nu V^\mu + V^\mu \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\mu} \tag{31}$$

So, (25) transforms as

$$\bar{\partial}_\alpha \bar{V}^\beta - \bar{\partial}_\beta \bar{V}^\alpha = \left(\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial y^\beta}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial^2 y^\beta}{\partial x^\mu \partial x^\nu} V^\nu \right) - \left(\frac{\partial x^\nu}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^\mu} \partial_\nu V^\mu + \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\mu} V^\mu \right) \tag{32}$$

$$\bar{\partial}_\alpha \bar{V}^\beta - \bar{\partial}_\beta \bar{V}^\alpha = \left(\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial y^\beta}{\partial x^\nu} \partial_\mu V^\nu - \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^\mu} \partial_\nu V^\mu \right) + \left(\frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial^2 y^\beta}{\partial x^\mu \partial x^\nu} V^\nu - \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\mu} V^\mu \right) \tag{33}$$

Once again the terms in the first parenthesis look okay. However, the terms in the second parenthesis screw it up similarly to that in B).

$\boxed{\text{Thus, these are not the components of a tensor.}}$

The 2nd derivative cannot become 0 in general, since the numerator is a partial in the y chart, while the denominator is in the x chart.

Now, we want to check if

$$\partial_\mu V_\nu - \partial_\nu V_\mu = \frac{\partial}{\partial x^\mu} V_\nu - \frac{\partial}{\partial x^\nu} V_\mu \tag{34}$$

are the components of a tensor.

Let's first focus on $\partial_\mu V_\nu$. The transformation of ∂_μ is shown in (20) and (21).

Now, transforming V_ν via (16), we consider this has a covariant index. Thus, this transforms like

$$\overline{V_\beta} = \frac{\partial x^\nu}{\partial y^\beta} V_\nu \tag{35}$$

Thus, $\partial_\mu V_\nu$ transforms as

$$\overline{\partial_\alpha} \, \overline{V_\beta} = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial y^\beta} V_\nu \right) \tag{36}$$

Using the product rule

$$\overline{\partial_\alpha} \, \overline{V_\beta} = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial}{\partial x^\mu}(V_\nu) + V_\nu \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial y^\beta}\right) \tag{37}$$

$$\overline{\partial_\alpha} \, \overline{V_\beta} = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \partial_\mu V_\nu + V_\nu \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial^2 x^\nu}{\partial x^\mu \partial y^\beta} \tag{38}$$

It then becomes easy to construct the transformation of $\partial_\nu V_\mu$ from this. It is

$$\overline{\partial_\beta} \, \overline{V_\alpha} = \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial x^\mu}{\partial y^\alpha} \partial_\nu V_\mu + V_\mu \frac{\partial x^\nu}{\partial y^\beta} \frac{\partial^2 x^\mu}{\partial x^\nu \partial y^\alpha} \tag{39}$$

So the transformation becomes

$$\overline{\partial_\alpha}\,\overline{V_\beta} - \overline{\partial_\beta}\,\overline{V_\alpha} = \begin{array}{l} \left(\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\partial_\mu V_\nu + V_\nu \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial^2 x^\nu}{\partial x^\mu \partial y^\beta}\right) \\ -\left(\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial x^\mu}{\partial y^\alpha}\partial_\nu V_\mu + V_\mu \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 x^\mu}{\partial x^\nu \partial y^\alpha}\right) \end{array} \tag{40}$$

Regrouping terms

$$\overline{\partial_\alpha}\,\overline{V_\beta} - \overline{\partial_\beta}\,\overline{V_\alpha} = \begin{array}{l} \left(\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\partial_\mu V_\nu - \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial x^\mu}{\partial y^\alpha}\partial_\nu V_\mu\right) \\ +\left(V_\nu \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial^2 x^\nu}{\partial x^\mu \partial y^\beta} - V_\mu \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 x^\mu}{\partial x^\nu \partial y^\alpha}\right) \end{array} \tag{41}$$

$$\overline{\partial_\alpha}\,\overline{V_\beta} - \overline{\partial_\beta}\,\overline{V_\alpha} = \begin{array}{l} \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right) \\ +\left(V_\nu \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial^2 x^\nu}{\partial x^\mu \partial y^\beta} - V_\mu \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 x^\mu}{\partial x^\nu \partial y^\alpha}\right) \end{array} \tag{42}$$

We consider that

$$\frac{\partial^2 x^\nu}{\partial x^\mu \partial y^\beta} = \frac{\partial}{\partial y^\beta}\left(\frac{\partial x^\nu}{\partial x^\mu}\right) \tag{43}$$

$$\frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu{}_\mu , \tag{44}$$

so

$$\frac{\partial^2 x^\nu}{\partial x^\mu \partial y^\beta} = 0 \tag{45}$$

Similarly then, $$\frac{\partial^2 x^\mu}{\partial x^\nu \partial y^\alpha} = 0 \tag{46}$$

So,

$$\overline{\partial}_\alpha \overline{V}_\beta - \overline{\partial}_\beta \overline{V}_\alpha = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \left(\partial_\mu V_\nu - \partial_\nu V_\mu \right) . \qquad (47)$$

$\boxed{\text{Thus, these are the components of a tensor.}}$

2)

(2) Let $f : M \to N$ be a differentiable map. Prove the following identities and write local coordinate expressions for them.

(a) $f^*(h\alpha) = (f^*h)(f^*\alpha)$, where $h \in C^\infty(N)$ and $\alpha \in T^*N$.

(b) $[f_*X, f_*Y] = f_*[X, Y]$, where $X, Y \in TM$.

A)

We let

$$f: M \to N \tag{1}$$

be a differentiable map.

With

$$h \in C^\infty(N) \tag{2}$$

and

$$\alpha \in T^*N, \tag{3}$$

we want to show

$$f^*(h\alpha) = (f^*h)(f^*\alpha). \tag{4}$$

We can represent f as

$$y^\beta = f^\beta(x^A), \tag{5}$$

wherein M has the chart $\{x^A\}$ and N $\{y^\beta\}$. (Chosen for convenience as they match the lecture)

We represent α as

$$\alpha = \alpha^\beta \, dy^\beta. \tag{6}$$

Thus, we find via transformation, via pullback

$$f^*\alpha = \left(\alpha_\beta \frac{\partial y^\beta}{\partial x^A}\right) dx^A \tag{7}$$

We represent h as

$$h = h(y^\beta) \quad (8)$$

Then, the pullback acts on h as

$$f^*h = h \circ f \quad (9)$$

$$f^*h = h(y^\beta(x^A)). \quad (10)$$

Thus,

$$(f^*h)(f^*\alpha) = (h(y^\beta(x^A)))\left(\alpha_\beta \frac{\partial y^\beta}{\partial x^A}\right) dx^A. \quad (11)$$

Now, for $f^*(h\alpha)$, note that $h\alpha$ is a scalar multiplication. Thus, we expect a similar transformation to (7).

$$f^*(h\alpha) = \left(h\alpha_\beta \frac{\partial y^\beta}{\partial x^A}\right) dx^A. \quad (12)$$

Since this is in the $\{x^A\}$ chart, we expand h in terms of this considering h is originally $\in C^\infty(N)$. Thus, we get

$$f^*(h\alpha) = \left((h(y^\beta(x^A)))\alpha_\beta \frac{\partial y^\beta}{\partial x^A}\right) dx^A \quad (13)$$

$$\boxed{f^*(h\alpha) = (h(y^\beta(x^A)))\left(\alpha_\beta \frac{\partial y^\beta}{\partial x^A}\right) dx^A} \quad (14)$$

(14) and (11) are identical, thus

$$\boxed{f^*(h\alpha) = (f^*h)(f^*\alpha)} \quad (15)$$

(14) is already a local coordinate expression.

B) For simplicity, we use the same charts for M and N.

We consider

$$X \in TM \tag{16}$$

$$Y \in TM. \tag{17}$$

We have X and Y represented as

$$X = X^A \partial_A \tag{18}$$

$$Y = Y^F \partial_F \tag{19}$$

We find that the push forward works as

$$f_* X = \left(X^A \frac{\partial y^\beta}{\partial x^A}\right) \partial_\beta \tag{20}$$

$$f_* Y = \left(Y^F \frac{\partial y^\gamma}{\partial x^F}\right) \partial_\gamma . \tag{21}$$

Since these are both vectors $\in TN$ now, we expect the commutator to work as

$$[f_* X, f_* Y] = \left((f_* X)^\beta \partial_\beta (f_* Y)^\gamma - (f_* Y)^\beta \partial_\beta (f_* X)^\gamma\right) \partial_\gamma \tag{22}$$

We now work on expanding this

$$(f_* X)^\beta \partial_\beta (f_* Y)^\gamma = \left(X^A \frac{\partial y^\beta}{\partial x^A}\right) \partial_\beta \left(Y^F \frac{\partial y^\gamma}{\partial x^F}\right) \tag{23}$$

$$\left(X^A \frac{\partial y^\beta}{\partial x^A}\right) \partial_\beta \left(Y^F \frac{\partial y^\gamma}{\partial x^F}\right) = \left(X^A \frac{\partial y^\beta}{\partial x^A}\right) \frac{\partial}{\partial y^\beta} \left(Y^F \frac{\partial y^\gamma}{\partial x^F}\right) \tag{24}$$

By product rule,

$$\frac{\partial}{\partial y^\beta}\left(Y^F \frac{\partial y^\gamma}{\partial x^F}\right) = \frac{\partial Y^F}{\partial y^\beta}\frac{\partial y^\gamma}{\partial x^F} + Y^F \frac{\partial^2 y^\gamma}{\partial y^\beta \partial x^F} \quad . \tag{25}$$

Thus,

$$(f_*X)^\beta \partial_\beta (f_*Y)^\gamma = X^A \frac{\partial y^\beta}{\partial x^A}\left(\frac{\partial Y^F}{\partial y^\beta}\frac{\partial y^\gamma}{\partial x^F} + Y^F \frac{\partial^2 y^\gamma}{\partial y^\beta \partial x^F}\right) . \tag{26}$$

So, we expect

$$(f_*Y)^\beta \partial_\beta (f_*X)^\gamma = Y^F \frac{\partial y^\beta}{\partial x^F}\left(\frac{\partial X^A}{\partial y^\beta}\frac{\partial y^\gamma}{\partial x^A} + X^A \frac{\partial^2 y^\gamma}{\partial y^\beta \partial x^A}\right) . \tag{27}$$

We evaluate the 2nd derivative as

$$\frac{\partial^2 y^\gamma}{\partial y^\beta \partial x^F} = \frac{\partial}{\partial x^F}\left(\frac{\partial y^\gamma}{\partial y^\beta}\right) \tag{28}$$

$$\frac{\partial^2 y^\gamma}{\partial y^\beta \partial x^F} = \frac{\partial}{\partial x^F}\left(\delta^\gamma_{\ \beta}\right), \tag{29}$$

$$\frac{\partial^2 y^\gamma}{\partial y^\beta \partial x^F} = 0 \tag{30}$$

which evaluates to 0, since the Kronecker delta is a constant.

Similarly,

$$\frac{\partial^2 y^\gamma}{\partial y^\beta \partial x^A} = 0 . \tag{31}$$

So we get,

$$(f_*X)^\beta \partial_\beta (f_*Y)^\gamma = X^A \frac{\partial y^\beta}{\partial x^A}\frac{\partial y^\gamma}{\partial x^F}\frac{\partial Y^F}{\partial y^\beta} \tag{32}$$

$$(f_*Y)^\beta \partial_\beta (f_*X)^\gamma = Y^F \frac{\partial y^\beta}{\partial x^F}\frac{\partial y^\gamma}{\partial x^A}\frac{\partial X^A}{\partial y^\beta} \quad . \tag{33}$$

Thus, (22) becomes

$$[f_* X, f_* Y] = \left(X^A \frac{\partial y^\beta}{\partial x^A} \frac{\partial y^\gamma}{\partial x^F} \frac{\partial Y^F}{\partial y^\beta} - Y^F \frac{\partial y^\beta}{\partial x^F} \frac{\partial y^\gamma}{\partial x^A} \frac{\partial X^A}{\partial y^\beta} \right) \partial_\gamma \quad (34)$$

Considering dummy indices and reindexing the second term

$$[f_* X, f_* Y] = \left(X^A \frac{\partial y^\beta}{\partial x^A} \frac{\partial y^\gamma}{\partial x^F} \frac{\partial Y^F}{\partial y^\beta} - Y^A \frac{\partial y^\beta}{\partial x^A} \frac{\partial y^\gamma}{\partial x^F} \frac{\partial X^F}{\partial y^\beta} \right) \partial_\gamma \quad (35)$$

Factoring this,

$$[f_* X, f_* Y] = \frac{\partial y^\beta}{\partial x^A} \frac{\partial y^\gamma}{\partial x^F} \left(X^A \frac{\partial Y^F}{\partial y^\beta} - Y^A \frac{\partial X^F}{\partial y^\beta} \right) \partial_\gamma \quad (36)$$

We now work on the RHS: $f_*[X, Y]$

We note that

$$[X, Y] \in TM. \quad (37)$$

And it gives

$$[X, Y] = \left(X^A \partial_A Y^F - Y^A \partial_A X^F \right) \partial_F \quad (38)$$

A push forward on this, considering both A and F indices then gives

$$f_*[X, Y] = \frac{\partial y^\beta}{\partial x^A} \frac{\partial y^\gamma}{\partial x^F} \left(X^A \partial_\beta Y^F - Y^A \partial_\beta X^F \right) \partial_\gamma . \quad (39)$$

$$f_*[X, Y] = \frac{\partial y^\beta}{\partial x^A} \frac{\partial y^\gamma}{\partial x^F} \left(X^A \frac{\partial Y^F}{\partial y^\beta} - Y^A \frac{\partial X^F}{\partial y^\beta} \right) \partial_\gamma . \quad (40)$$

Thus, indeed

$$\boxed{[f_* X, f_* Y] = f_*[X, Y]} \quad (41)$$

3)

(3) Consider the map $\phi : M(\subset \mathbb{R}^2) \to \mathbb{R}^3$ given by

$$\phi(u,v) = \left(2\cos u + v\sin\frac{u}{2}\cos u, 2\sin u + v\sin\frac{u}{2}\sin u, v\cos\frac{u}{2}\right) =: (x(u,v), y(u,v), z(u,v)),$$

where $0 \le u < 2\pi$ and $-1 < v < 1$. (a) Plot the image $\phi(M)$ in $\mathbb{R}^3$. What surface is this? (b) If $g = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ is the Euclidean metric on $\mathbb{R}^3$, compute $\phi^* g$. What does $\phi^* g$ mean geometrically?

We have the map

$$\phi : M(\subset \mathbb{R}^2) \to \mathbb{R}^3, \quad (1)$$

which is given by

$$\phi(u,v) = \left(2\cos(u) + v\sin\left(\frac{u}{2}\right)\cos(u), 2\sin(u) + v\sin\left(\frac{u}{2}\right)\sin(u), v\cos\left(\frac{u}{2}\right)\right) \quad (2)$$

$$\phi(u,v) =: (x(u,v), y(u,v), z(u,v)). \quad (3)$$

In this case, u and v are bounded as

$$0 \le u < 2\pi, \quad (4)$$

and

$$-1 < v < 1. \quad (5)$$

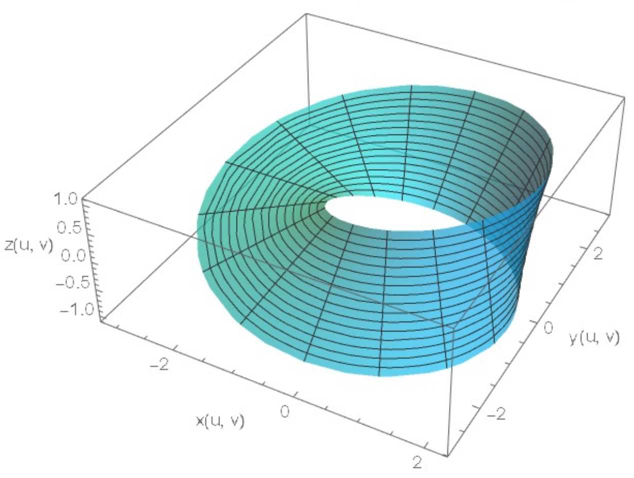
A) We want to plot the image

$$\phi(M) \subset \mathbb{R}^3 \quad (6)$$

Using Mathematica, we see that this is the Möbius Strip.

```
In[9]:= ParametricPlot3D[{2 Cos[u] + v Sin[u/2] Cos[u],
  2 Sin[u] + v Sin[u/2] Sin[u], v Cos[u/2]}, {u, 0, 2 π},
 {v, -1, 1},
 PlotStyle → Directive[RGBColor[0.16, 0.9, 1.], Opacity[0.72]],
 AxesLabel → {"x(u, v)", "y(u, v)", "z(u, v)"}]
```

Out[9]=



B) The Euclidean Metric in $\mathbb{R}^3$ is

$$g = dx^2 + dy^2 + dz^2. \tag{7}$$

We want to compute the pullback operation $\phi^* g$

We know that

$$g : TN \times TN \to \mathbb{R}, \tag{8}$$

so

$$\phi^* g : TM \times TM \to \mathbb{R}. \tag{9}$$

Considering M has the chart

$$\{x^A\} = \{u, v\}, \tag{10}$$

and $\mathbb{R}^3$ has the chart

$$\{y^\alpha\} = \{x, y, z\}, \tag{11}$$

we apply the formula derived in class

$$\phi^* g = \left(g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^A} \frac{\partial y^\beta}{\partial x^B} \right) dx^A \otimes dx^B. \tag{12}$$

So we now calculate

$$\frac{\partial x}{\partial u} = -2\sin(u) + v\left(\tfrac{1}{2}\cos\left(\tfrac{u}{2}\right)\cos(u) - \sin\left(\tfrac{u}{2}\right)\sin(u)\right) \tag{13}$$

$$\frac{\partial x}{\partial v} = \cos(u)\sin\left(\tfrac{u}{2}\right) \tag{14}$$

$$\frac{\partial y}{\partial u} = 2\cos(u) + v\left(\tfrac{1}{2}\cos\left(\tfrac{u}{2}\right)\sin(u) + \cos(u)\sin\left(\tfrac{u}{2}\right)\right) \tag{15}$$

$$\frac{\partial y}{\partial r} = \sin\left(\frac{u}{2}\right)\sin(u) \tag{16}$$

$$\frac{\partial z}{\partial u} = -\frac{1}{2} r \sin\left(\frac{u}{2}\right) \tag{17}$$

$$\frac{\partial z}{\partial r} = \cos\left(\frac{u}{2}\right) \tag{18}$$

From (7) we know that

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{19}$$

Thus, we only have to consider the diagonal terms of (19) in (12). Therefore, the components are

$$(\phi^* g)_{uu} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial u} \tag{20}$$

$$(\phi^* g)_{rr} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial r} \tag{21}$$

$$(\phi^* g)_{ur} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial r} \tag{22}$$

$$(\phi^* g)_{ru} = \frac{\partial x}{\partial r}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial u} . \tag{23}$$

(22) and (23) are clearly equivalent, so

$$(\phi^* g)_{ur} = (\phi^* g)_{ru} . \tag{24}$$

Using Mathematica, these evaluate to

$$(\phi^* g)_{uu} = 4 + \frac{3}{4} v^2 - \frac{1}{2} v^2 \cos(u) + 4 v \sin\left(\frac{u}{2}\right) \quad (25)$$

$$(\phi^* g)_{vv} = 1 \quad (26)$$

$$(\phi^* g)_{uv} = 0 \quad (27)$$

$$(\phi^* g)_{vv} = 0 \quad (28)$$

Thus,

$$\boxed{\phi^* g = \left(4 + v^2\left(\frac{3}{4} - \frac{1}{2}\cos(u)\right) + 4 v \sin\left(\frac{u}{2}\right)\right) du^2 + dv^2} \quad (29)$$

Geometrically, $\phi^* g$ is interpreted as an induced metric from N to M. In our case this was from the Cartesian chart on $\mathbb{R}^3$ $\{x, y, z\}$ to the chart $\{u, v\}$ on $M \subset \mathbb{R}^2$.

4) (4) Let $g = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$ be the metric on the two-sphere in spherical coordinates. Verify that if $K = \partial_\phi$, then $\pounds_K g = 0$. Find another vector field on the two-sphere that satisfies this. Sketch this vector field on the two-sphere.

We let

$$g = d\theta^2 + \sin^2\theta \, d\phi^2 \tag{1}$$

be the metric on the two-sphere in spherical coordinates.

We have the Killing vector

$$K = \partial_\phi \tag{2}$$

We want to show if

$$\pounds_K g = 0. \tag{3}$$

The components of $\pounds_K g$ are given by

$$(\pounds_K g)_{\mu\nu} = g_{\mu\nu,\rho} K^\rho + g_{\rho\nu} K^\rho_{,\mu} + g_{\mu\rho} K^\rho_{,\nu} \tag{4}$$

For (2), since all its components are constant.

$$\partial_{\phi,\mu} = 0 \tag{5}$$

$$\partial_{\phi,\nu} = 0 \tag{6}$$

So

$$(\pounds_{\partial_\phi} g)_{\mu\nu} = g_{\mu\nu,\rho} (\partial_\phi)^\rho \tag{7}$$

From (1), off-diagonal elements of g are 0

$$g_{\phi\theta} = 0 \tag{8}$$

$$g_{\theta\phi} = 0 \tag{9}$$

$$\left(\mathcal{L}_{\partial_\phi} g\right)_{\theta\theta} = \left(\frac{\partial}{\partial\theta}(1)\right)(0) + \left(\frac{\partial}{\partial\phi}(1)\right)(1) \tag{10}$$

$$\left(\mathcal{L}_{\partial_\phi} g\right)_{\theta\theta} = 0 \tag{11}$$

$$\left(\mathcal{L}_{\partial_\phi} g\right)_{\phi\phi} = \left(\frac{\partial}{\partial\theta}(\sin^2\theta)\right)(0) + \left(\frac{\partial}{\partial\phi}(\sin^2\theta)\right)(1) \tag{12}$$

$$\left(\mathcal{L}_{\partial_\phi} g\right)_{\phi\phi} = 0 \tag{13}$$

Thus, by (8), (9), (11), and (12), indeed

$$\boxed{\mathcal{L}_{\partial_\phi} g = 0} \tag{14}$$

Finding another applicable vector field amounts to solving (3).

Since a two-sphere is constant in r, we only have to consider θ, ϕ.

So we have to solve (4) as

$$\left(\mathcal{L}_K g\right)_{\theta\theta} = 0 \tag{15}$$

$$\left(\mathcal{L}_K g\right)_{\theta\phi} = 0 \tag{16}$$

$$\left(\mathcal{L}_K g\right)_{\phi\theta} = 0 \tag{17}$$

$$\left(\mathcal{L}_K g\right)_{\phi\phi} = 0 \tag{18}$$

By symmetry,

$$(\pounds_K g)_{\theta\phi} = (\pounds_K g)_{\phi\theta}. \tag{19}$$

So, we only have three equations to solve.

(15) becomes

$$\frac{\partial g_{\theta\theta}}{\partial\theta} K^\theta + \frac{\partial g_{\theta\theta}}{\partial\phi} K^\phi + g_{\theta\theta}\frac{\partial K^\theta}{\partial\theta} + g_{\phi\theta}\frac{\partial K^\phi}{\partial\theta} + g_{\theta\theta}\frac{\partial K^\theta}{\partial\theta} + g_{\theta\phi}\frac{\partial K^\phi}{\partial\theta} = 0 \tag{20}$$

Considering (8) and (9), this becomes

$$\frac{\partial g_{\theta\theta}}{\partial\theta} K^\theta + \frac{\partial g_{\theta\theta}}{\partial\phi} K^\phi + 2 g_{\theta\theta}\frac{\partial K^\theta}{\partial\theta} = 0 \tag{21}$$

Plugging in, this becomes

$$\frac{\partial}{\partial\theta}[1] K^\theta + \frac{\partial}{\partial\phi}[1] K^\phi + 2(1)\frac{\partial K^\theta}{\partial\theta} = 0. \tag{22}$$

$$2\frac{\partial K^\theta}{\partial\theta} = 0. \tag{23}$$

The condition may be simplified as

$$\frac{\partial K^\theta}{\partial\theta} = 0. \tag{24}$$

(16) becomes

$$\frac{\partial g_{\theta\phi}}{\partial\theta} K^\theta + \frac{\partial g_{\theta\phi}}{\partial\phi} K^\phi + g_{\theta\phi}\frac{\partial K^\theta}{\partial\theta} + g_{\phi\phi}\frac{\partial K^\phi}{\partial\theta} + g_{\theta\theta}\frac{\partial K^\theta}{\partial\phi} + g_{\theta\phi}\frac{\partial K^\phi}{\partial\phi} = 0 \tag{25}$$

Considering (8) and (9), this becomes

$$g_{\phi\phi}\frac{\partial K^{\phi}}{\partial\theta} + g_{\theta\theta}\frac{\partial K^{\theta}}{\partial\phi} = 0 \qquad (26)$$

Plugging in, we get

$$\sin^2\theta\frac{\partial K^{\phi}}{\partial\theta} + \frac{\partial K^{\theta}}{\partial\phi} = 0 \qquad (27)$$

(18) becomes

$$\frac{\partial g_{\phi\phi}}{\partial\theta}K^{\theta} + \frac{\partial g_{\phi\phi}}{\partial\phi}K^{\phi} + g_{\theta\phi}\frac{\partial K^{\theta}}{\partial\phi} + g_{\phi\phi}\frac{\partial K^{\phi}}{\partial\phi} + g_{\phi\theta}\frac{\partial K^{\theta}}{\partial\phi} + g_{\phi\phi}\frac{\partial K^{\phi}}{\partial\phi} = 0 \qquad (28)$$

Considering (8) and (9) this becomes

$$\frac{\partial g_{\phi\phi}}{\partial\theta}K^{\theta} + \frac{\partial g_{\phi\phi}}{\partial\phi}K^{\phi} + g_{\phi\phi}\frac{\partial K^{\phi}}{\partial\phi} + g_{\phi\phi}\frac{\partial K^{\phi}}{\partial\phi} = 0 \qquad (29)$$

Plugging in, this becomes

$$\frac{\partial}{\partial\theta}\left[\sin^2\theta\right]K^{\theta} + \frac{\partial}{\partial\phi}\left[\sin^2\theta\right]K^{\phi} + 2\sin^2\theta\frac{\partial K^{\phi}}{\partial\phi} = 0 \qquad (30)$$

$$2\sin\theta\cos\theta\, K^{\theta} + 2\sin^2\theta\frac{\partial K^{\phi}}{\partial\phi} = 0 \qquad (31)$$

Dividing by $2\sin\theta$, we get

$$\cos\theta\, K^{\theta} + \sin\theta\frac{\partial K^{\phi}}{\partial\phi} = 0 \qquad (32)$$

So our conditions are (24), (27), and (32)

(24) tells us that

$$K^\theta = K^\theta(\phi), \tag{33}$$

since it can't have θ dependence.

(27) and (32) can be rewritten as

$$\sin^2\theta \frac{\partial K^\phi}{\partial \theta} = -\frac{\partial K^\theta}{\partial \phi} \tag{34}$$

$$\cos\theta \, K^\theta = -\sin\theta \frac{\partial K^\phi}{\partial \phi} \tag{35}$$

(33) and (35) yield

$$K^\theta(\phi) = -\tan\theta \frac{\partial K^\phi}{\partial \phi} \tag{36}$$

Rearranging and integrating, this becomes

$$-\cot\theta \int K^\theta(\phi)\,d\phi = \int K^\phi \, dK^\phi \tag{37}$$

$$K^\phi = -\cot\theta \int K^\theta(\phi)\,d\phi. \tag{38}$$

Using (38) in (34), this gives

$$\sin^2\theta \frac{\partial}{\partial\theta}\left(-\cot\theta \int K^\theta(\phi)\,d\phi\right) = -\frac{\partial K^\theta}{\partial \phi} \tag{39}$$

$$\sin^2\theta\left(\csc^2\theta \int K^\theta(\phi)\,d\phi\right) = -\frac{\partial K^\theta}{\partial \phi} \tag{40}$$

$$\int K^\theta(\phi)\,d\phi = -\frac{\partial K^\theta}{\partial \phi} \tag{41}$$

(41) gives

$$\iint K^{\theta}(\phi)\,d\phi\,d\phi = -K^{\theta}(\phi). \tag{42}$$

Solving (42) is of course equivalent to

$$-\frac{\partial^2(K^{\theta}(\phi))}{\partial\phi^2} = K^{\theta}(\phi), \tag{43}$$

$$-K^{\theta}(\phi) = \frac{\partial^2(K^{\theta}(\phi))}{\partial\phi^2} \tag{44}$$

when we differentiate twice, and move the negative sign.

The solution to this DE is obviously

$$K^{\theta}(\phi) = A\sin\phi + B\cos\phi, \tag{45}$$

where A and B are some constants.

(38) then yields

$$K^{\phi} = -\cot\theta \int (A\sin\phi + B\cos\phi)\,d\phi \tag{46}$$

$$K^{\phi} = A\cos\phi\cot\theta - B\sin\phi\cot\theta \tag{47}$$

Since the Killing vectors in this case take the form

$$K = K^{\theta}\partial_{\theta} + K^{\phi}\partial_{\phi}, \tag{48}$$

we have

$$\boxed{K = (A\sin\phi + B\cos\phi)\partial_\theta + (A\cos\phi\cot\theta - B\sin\phi\cot\theta)\partial_\phi} \qquad (49)$$

In the interest of plotting, we assume

$$A = 1 \qquad (50)$$
$$B = 1, \qquad (51)$$

and that the radius of the two-sphere

$$R = 1. \qquad (52)$$

So that

$$K = (\sin\phi + \cos\phi)\partial_\theta + (\cos\phi\cot\theta - \sin\phi\cot\theta)\partial_\phi \qquad (53)$$

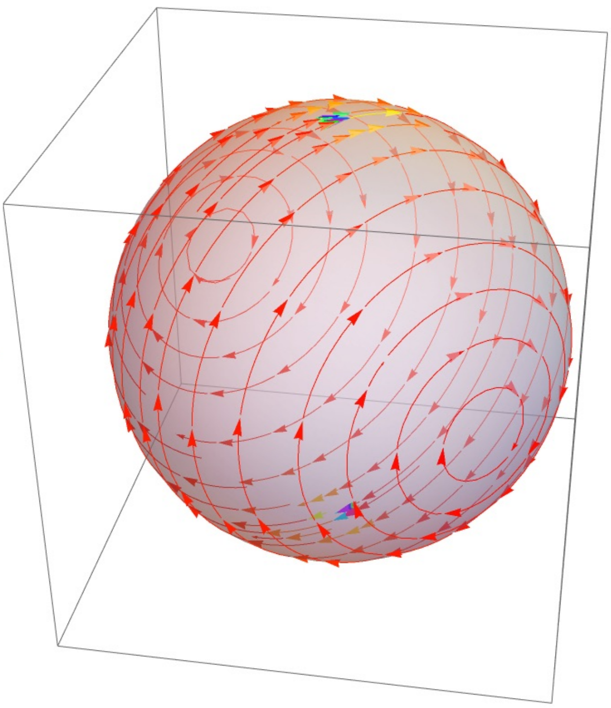


Figure 1. Plot of vector field K on unit two-sphere.

Miguel Yulo 2018-04845 Physics 225 SET 3

I would like to acknowledge the help provided by Luke Gurrea, JR Oidem, and Val Balagon in my answering of this problem set.

I however certify that the following work is from my own efforts

Miguel Yulo

1) ~~(1) Prove the following identity~~

$$\nabla_a \left(R^e{}_{bcd} \omega_e \right) = \omega_e \nabla_a R^e{}_{bcd} + R^e{}_{bcd} \nabla_a \omega_e$$

using index-free notation.

We seek to prove

$$\nabla_a (R^e{}_{bcd} \omega_e) = \omega_e \nabla_a R^e{}_{bcd} + R^e{}_{bcd} \nabla_a \omega_e \,. \qquad (1)$$

We note that in index-free notation

$$R^e{}_{bcd} = R(dx^e, \partial_b, \partial_c, \partial_d) \qquad (2)$$

$$R^e{}_{bcd} = dx^e(R(\partial_b, \partial_c), \partial_d), \qquad (3)$$

which in terms of a tensor product is

$$R^e{}_{bcd} = CCCC(dx^e \otimes R \otimes \partial_b \otimes \partial_c \otimes \partial_d) \,. \qquad (4)$$

We can then interpret $R^e{}_{bcd} \omega_e$ as $R^e{}_{bcd}$ acting on a one-form

$$R^e{}_{bcd} \omega_e = CCCCC(\omega \otimes R \otimes \partial_b \otimes \partial_c \otimes \partial_d) \,. \qquad (5)$$

Thus, the covariant derivative acts on this as

$$\nabla_w (CCCCC(\omega \otimes R \otimes \partial_b \otimes \partial_c \otimes \partial_d)) = CCCCC(\nabla_w(\omega \otimes R \otimes \partial_b \otimes \partial_c \otimes \partial_d)) \cdot (6)$$

We can expand this as

$$CCCCC\left(\nabla_w(\omega\otimes R\otimes\partial_b\otimes\partial_c\otimes\partial_d)\right)=CCCCC\begin{pmatrix}(\nabla_w\omega)\otimes R\otimes\partial_b\otimes\partial_c\otimes\partial_d+\\ \omega\otimes\nabla_w R\otimes\partial_b\otimes\partial_c\otimes\partial_d+\\ \omega\otimes R\otimes\nabla_w\partial_b\otimes\partial_c\otimes\partial_d+\\ \omega\otimes R\otimes\partial_b\otimes\nabla_w\partial_c\otimes\partial_d+\\ \omega\otimes R\otimes\partial_b\otimes\partial_c\otimes\nabla_w\partial_d\end{pmatrix}. \tag{7}$$

Considering that for a metric-compatible connection, which we assume, the covariant derivatives of vectors vanish

$$\nabla_w\partial_b=0 \tag{8}$$
$$\nabla_w\partial_c=0 \tag{9}$$
$$\nabla_w\partial_d=0\,. \tag{10}$$

Thus, (7) simplifies to

$$CCCCC\left(\nabla_w(\omega\otimes R\otimes\partial_b\otimes\partial_c\otimes\partial_d)\right)=CCCCC\begin{pmatrix}(\nabla_w\omega)\otimes R\otimes\partial_b\otimes\partial_c\otimes\partial_d+\\ \omega\otimes\nabla_w R\otimes\partial_b\otimes\partial_c\otimes\partial_d\end{pmatrix}. \tag{11}$$

Translating these terms back into abstract index notation gives

$$CCCCC\left((\nabla_w\omega)\otimes R\otimes\partial_b\otimes\partial_c\otimes\partial_d\right)=R^e{}_{bcd}(\nabla_a\omega_e) \tag{12}$$
$$CCCCC\left(\omega\otimes\nabla_w R\otimes\partial_b\otimes\partial_c\otimes\partial_d\right)=\omega_e(\nabla_a R^e{}_{bcd})\,, \tag{13}$$

where we have taken the covariant derivative as ∇_a.

Thus, adding these together and using (5), we get

$$\boxed{\nabla_a(R^e{}_{bcd}\,\omega_e)=\omega_e\nabla_a R^e{}_{bcd}+R^e{}_{bcd}\nabla_a\omega_e}\,. \tag{14}$$

2)

(2) Consider the manifold S^2 with the usual metric

$$g = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2.$$

(i) Compute the Christoffel symbols of the Levi-Civita connection.

(ii) Write down the equations of parallel transport for a vector V on a general curve $\{\theta(t), \phi(t)\}$.

(iii) Specialize these to curves of constant latitude (say $\{\theta(t) = \theta_0, \phi(t) = t\}$) and constant longitude $\{\theta(t) = t, \phi(t) = \phi_0\}$. How does V change when parallel-transported along a right spherical triangle (starting from the north pole, down a constant-longitude curve up to the equator, east along a constant-latitude curve, and again up a constant-longitude curve back to the north pole)?

We have the metric

$$g = d\theta^2 + \sin^2\theta \, d\phi^2 . \qquad (1)$$

i) We can calculate the Christoffel Symbols from the Euler-Lagrange equations we use for geodesics.

Consider that in general, the geodesic equations take the form

$$\ddot{x}^\sigma + \Gamma^\sigma_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 . \qquad (2)$$

We take the Lagrangian as

$$L = g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta . \qquad (3)$$

It can be seen that the metric in (1) is diagonal, so we expect two geodesic equations.

Reading off the metric, (3) gives

$$L = \dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 . \qquad (4)$$

Where the $\cdot$ represents a derivative wrt. our parameter t.

Starting with the θ equation, we require

$$\frac{\partial L}{\partial t}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0, \tag{5}$$

where in

$$\frac{\partial L}{\partial \theta} = 2 \sin\theta \cos\theta\, \dot{\phi}^2, \tag{6}$$

$$\frac{\partial L}{\partial t}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = \frac{\partial}{\partial \lambda}\left[2\, \dot{\theta}\right], \tag{7}$$

and

$$\frac{\partial L}{\partial t}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = 2\, \ddot{\theta}, \tag{8}$$

Therefore, the geodesic equation reads as

$$2\ddot{\theta} - 2 \sin\theta \cos\theta\, \dot{\phi}^2 = 0 . \tag{9}$$

In the form of (2), this simplifies to

$$\ddot{\theta} - \sin\theta \cos\theta\, \dot{\phi}^2 = 0 . \tag{10}$$

Now, for the ϕ equation, we require

$$\frac{\partial L}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}}\right] - \frac{\partial L}{\partial \phi} = 0, \tag{11}$$

wherein

$$\frac{\partial L}{\partial \phi} = 0, \qquad (12)$$

$$\frac{\partial L}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}}\right] = \frac{\partial}{\partial \lambda}\left[2\sin^2\theta\, \dot{\phi}\right], \qquad (13)$$

and

$$\frac{\partial L}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}}\right] = 2\sin^2\theta\, \ddot{\phi} + 4\sin\theta\cos\theta\, \dot{\theta}\, \dot{\phi}. \qquad (14)$$

Therefore, the geodesic equation reads as

$$2\sin^2\theta\, \ddot{\phi} + 4\sin\theta\cos\theta\, \dot{\theta}\dot{\phi} = 0. \qquad (15)$$

In the form of (2), this simplifies to

$$\ddot{\phi} + 2\cot\theta\, \dot{\theta}\dot{\phi} = 0. \qquad (16)$$

Using (2), we can read off (10) to find

$$\boxed{\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta}. \qquad (17)$$

Reading off (16),

$$2\cot\theta = \Gamma^{\phi}_{\theta\phi} + \Gamma^{\phi}_{\phi\theta}. \qquad (18)$$

Since

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta}, \qquad (19)$$

$$\boxed{\begin{aligned}\Gamma^{\phi}_{\theta\phi} &= \cot\theta \qquad (20)\\ \Gamma^{\phi}_{\phi\theta} &= \cot\theta \qquad (21)\end{aligned}}\ .$$

All other Christoffel symbols are 0.

ii) For a vector V, the equations of parallel transport are given by

$$\frac{DV}{dt} = 0 \tag{22}$$

$$\left(\frac{dV^{\sigma}}{dt} + \Gamma^{\sigma}_{\alpha\beta} V^{\alpha}\dot{x}^{\beta}\right)\partial_{\sigma} = 0\,. \tag{23}$$

Thus,

$$\frac{dV^{\sigma}}{dt} + \Gamma^{\sigma}_{\alpha\beta} V^{\alpha}\dot{x}^{\beta} = 0\,. \tag{24}$$

For our only non-zero Christoffels then, we have for $\sigma = \theta$

$$\frac{dV^{\theta}}{dt} + \Gamma^{\theta}_{\phi\phi} V^{\phi}\dot{\phi} = 0 \tag{25}$$

$$\boxed{\left(\frac{dV^{\theta}}{dt} - V^{\phi}\sin\theta\cos\theta\,\dot{\phi}\right)\partial_{\theta} = 0}\,. \tag{26}$$

For $\sigma = \phi$, we have

$$\frac{dV^\phi}{dt} + \Gamma^\phi_{\theta\phi} V^\theta \dot{\phi} + \Gamma^\phi_{\phi\theta} V^\phi \dot{\theta} = 0 \tag{27}$$

$$\boxed{\left(\frac{dV^\phi}{dt} + \cot\theta\left(V^\theta \dot{\phi} + V^\phi \dot{\theta}\right)\right)\partial_\phi = 0} . \tag{28}$$

iii) For constant latitude

$$\theta(t) = \theta_0 , \tag{29}$$

(26) gives

$$\frac{dV^\theta}{dt} - V^\phi \sin\theta_0 \cos\theta_0 \dot{\phi} = 0 \tag{30}$$

$$\boxed{\frac{dV^\theta}{dt} = V^\phi \sin\theta_0 \cos\theta_0 \dot{\phi}} . \tag{31}$$

(28) gives

$$\frac{dV^\phi}{dt} + \cot\theta\left(V^\theta \dot{\phi} + V^\phi (0)\right) \tag{32}$$

$$\boxed{\frac{dV^\phi}{dt} = -\cot\theta_0 \dot{\phi} V^\theta} . \tag{33}$$

Now for constant longitude, where

$$\phi(t) = \phi_0 . \tag{34}$$

(26) gives

$$\frac{dV^\theta}{dt} = V^\phi \sin\theta\cos\theta(0) \tag{35}$$

$$\frac{dV^\theta}{dt} = 0 . \tag{36}$$

Thus,

$$\boxed{V^\theta = C,} \tag{37}$$

where C is some constant.

(28) gives

$$\frac{dV^\phi}{dt} = -\cot\theta\left(V^\theta(0) + V^\phi\dot{\theta}\right) \tag{38}$$

$$\boxed{\frac{dV^\phi}{dt} = -\cot\theta\, V^\phi\dot{\theta}} . \tag{39}$$

For transporting a vector around a right spherical triangle, let's consider

$$P = (\theta, \phi) \tag{40}$$

$$P_1 = (\epsilon, 0) \tag{41}$$

$$P_2 = (\tfrac{\pi}{2}, 0) \tag{42}$$

$$P_3 = (\tfrac{\pi}{2}, \chi) \,. \tag{43}$$

and back to the North Pole

$$P_4 = (\epsilon, \chi) \tag{44}$$

$$P_5 = (\epsilon, 0) \,. \tag{45}$$

So at P_1 we just have

$$V_{P_1} = V_{P_1}^{\phi} \partial_\phi + V_{P_1}^{\theta} \partial_\theta \,. \tag{46}$$

From $\underline{P_1 \to P_2,}$ this is constant longitude. And

$$\dot{\theta} = 1 \,. \tag{47}$$

So from (37)

$$V_{P_2}^{\theta} = V_{P_1}^{\theta} , \tag{48}$$

and (39) gives

$$\frac{dV^\phi}{dt} = -\cot\theta \, V^\phi (1) \tag{49}$$

Taking the chain rule, we can use

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}, \tag{50}$$

and considering (47)

$$\frac{dV^{\phi}}{d\theta} = -\cot\theta\, V^{\phi}. \tag{51}$$

Considering

$$\theta_{\text{initial}} = \epsilon, \tag{52}$$

$$V^{\theta}_{\text{initial}} = V^{\theta}(\theta = \epsilon) \tag{53}$$

$$V^{\phi}_{\text{initial}} = V^{\phi}(\theta = \epsilon) \tag{54}$$

Solving, we find

$$-\int \frac{dV^{\phi}}{V^{\phi}} = \int \cot\theta\, d\theta \tag{55}$$

$$\ln|-V^{\phi}| = \ln(\sin\theta) + D \tag{56}$$

$$V^{\phi} = V^{\phi}_{\text{initial}} \sin\theta. \tag{57}$$

Thus,

$$V^{\phi}_{P_2} = V^{\phi}_{P_1} \sin(\epsilon) \tag{58}$$

$$V_{P_2} = V_{P_1}{}^{\phi} \sin(\epsilon)\, \partial_\phi + V_{P_1}{}^{\theta}\, \partial_\theta \qquad (59)$$

Now, for $\underline{P_2 \to P_3}$, this is constant latitude.

Note that

$$\dot{\phi} = 1. \qquad (60)$$

From (31) and (33)

$$\frac{dV^\theta}{dt} = V^\phi \sin\left(\tfrac{\pi}{2}\right) \cos\left(\tfrac{\pi}{2}\right)(1) \qquad (61)$$

$$\frac{dV^\theta}{dt} = 0 \qquad (62)$$

$$\frac{dV^\phi}{dt} = -\cot\left(\tfrac{\pi}{2}\right) V^\theta \qquad (63)$$

$$\frac{dV^\phi}{dt} = 0. \qquad (64)$$

(62) and (64) imply

$$V^\theta = \text{Constant} \qquad (65)$$

$$V^\phi = \text{Constant}, \qquad (66)$$

so

$$V_{P_3}^\theta = V_{P_2}^\theta \qquad (67)$$

$$V_{P_3}^\phi = V_{P_2}^\phi \qquad (68)$$

$$V_{P_3}^{\theta} = V_{P_1}^{\theta} \tag{69}$$

$$V_{P_3}^{\phi} = V_{P_1}^{\phi} \sin(\epsilon) . \tag{70}$$

Now for $\underline{P_3 \to P_4}$, this is constant longitude, where

$$\dot{\theta} = -1 . \tag{71}$$

Similar to (48) and (58), we have

$$V_{P_4}^{\theta} = V_{P_3}^{\theta} \tag{72}$$

$$V_{P_4}^{\phi} = V_{P_3}^{\phi} \sin(\epsilon), \tag{73}$$

which give

$$V_{P_4}^{\theta} = V_{P_1}^{\theta} \tag{74}$$

$$V_{P_4}^{\phi} = V_{P_1}^{\phi} \sin^2(\epsilon) . \tag{75}$$

Now for $\underline{P_4 \to P_5}$, this is constant latitude, where

$$\dot{\phi} = -1 \tag{76}$$

From (31) and (33),

$$\frac{dV^{\theta}}{dt} = V^{\phi} \sin(\epsilon) \cos(\epsilon)(-1) \tag{77}$$

$$\frac{dV^{\theta}}{dt} = -V^{\phi} \sin(\epsilon) \cos(\epsilon) \tag{78}$$

$$\frac{dV^{\phi}}{dt} = -\cot(\epsilon)\, V^{\theta}(-1) \tag{79}$$

$$\frac{dV^{\phi}}{dt} = \cot(\epsilon)\, V^{\theta}. \tag{80}$$

Using chain rule,

$$\frac{dV^{\theta}}{d\phi}(-1) = -V^{\phi}\sin(\epsilon)\cos(\epsilon) \tag{81}$$

$$\frac{dV^{\theta}}{d\phi} = V^{\phi}\sin(\epsilon)\cos(\epsilon) \tag{82}$$

$$\frac{dV^{\phi}}{d\phi}(-1) = \cot(\epsilon)\, V^{\theta} \tag{83}$$

$$\frac{dV^{\phi}}{d\phi} = \cot(\epsilon)\, V^{\theta}. \tag{84}$$

Differentiating (84) by ϕ

$$\frac{d^2V^{\phi}}{d\phi^2} = -\cot(\epsilon)\, \frac{dV^{\theta}}{d\phi}. \tag{85}$$

Plugging in (82) gives

$$\frac{d^2V^{\phi}}{d\phi^2} = -\cot(\epsilon)\left(V^{\phi}\sin(\epsilon)\cos(\epsilon)\right) \tag{86}$$

$$\frac{d^2V^\phi}{d\phi^2} = -\cos^2(\epsilon)V^\phi. \tag{87}$$

This is solved as

$$V^\phi = A\cos(\cos(\epsilon)\phi) + B\sin(\cos(\epsilon)\phi). \tag{88}$$

Plugging this into (82) gives

$$\frac{dV^\theta}{d\phi} = \sin(\epsilon)\cos(\epsilon)\Big(A\cos(\cos(\epsilon)\phi) + B\sin(\cos(\epsilon)\phi)\Big). \tag{89}$$

Integrating (89) then we get

$$V^\theta = \sin(\epsilon)\cos(\epsilon)\Big(A\int\cos(\cos(\epsilon)\phi)d\phi + B\int\sin(\cos(\epsilon)\phi)\,d\phi\Big) \tag{90}$$

$$V^\theta = \sin(\epsilon)\Big(A\sin(\cos(\epsilon)\phi) - B\cos(\cos(\epsilon)\phi)\Big). \tag{91}$$

We apply the initial conditions of P_4 to this to give

$$V^\phi_{P_4} = A\cos(\cos(\epsilon)\chi) + B\sin(\cos(\epsilon)\chi) \tag{92}$$

$$V^\phi_{P_1}\sin^2(\epsilon) = A\cos(\cos(\epsilon)\chi) + B\sin(\cos(\epsilon)\chi) \tag{93}$$

$$V^\theta_{P_4} = \sin(\epsilon)\Big(A\sin(\cos(\epsilon)\chi) - B\cos(\cos(\epsilon)\chi)\Big) \tag{94}$$

$$V^\theta_{P_1} = \sin(\epsilon)\Big(A\sin(\cos(\epsilon)\chi) - B\cos(\cos(\epsilon)\chi)\Big) \tag{95}$$

For simplicity, let

$$\cos(\epsilon)\chi = k \tag{96}$$

$$V_{P_1}^{\phi}\sin^2(\epsilon) = C \tag{97}$$

$$\frac{V_{P_1}^{\theta}}{\sin(\epsilon)} = D \tag{99}$$

Solving for A and B, we find that

$$A = \frac{C}{\cos(k)} - B\frac{\sin(k)}{\cos(k)} \tag{100}$$

$$A = \frac{D}{\sin(k)} + B\frac{\cos(k)}{\sin(k)} \tag{101}$$

$$\frac{C}{\cos(k)} - B\frac{\sin(k)}{\cos(k)} = \frac{D}{\sin(k)} + B\frac{\cos(k)}{\sin(k)} \tag{102}$$

$$B\left(\frac{\sin(k)}{\cos(k)} + \frac{\cos(k)}{\sin(k)}\right) = \frac{C}{\cos(k)} - \frac{D}{\sin(k)} \tag{103}$$

$$B\left(\frac{1}{\sin(k)\cos(k)}\right) = \frac{C\sin(k) - D\cos(k)}{\sin(k)\cos(k)} \tag{104}$$

$$B = C\sin(k) - D\cos(k) \tag{105}$$

Plugging (105) into (100) gives

$$A = \frac{C}{\cos(k)} - \left(\left(C\sin(k) - D\cos(k)\right)\frac{\sin(k)}{\cos(k)}\right) \tag{106}$$

$$A = C\frac{(1-\sin^2(k))}{\cos(k)} + D\sin(k) \tag{107}$$

$$A = C\cos(k) + D\sin(k). \tag{108}$$

Therefore,

$$A = V_{P_1}^{\phi}\sin^2(\epsilon)\cos(\cos(\epsilon)\chi) + \frac{V_{P_1}^{\theta}}{\sin(\epsilon)}\sin(\cos(\epsilon)\chi) \tag{109}$$

$$B = V_{P_1}^{\phi}\sin^2(\epsilon)\sin(\cos(\epsilon)\chi) - \frac{V_{P_1}^{\theta}}{\sin(\epsilon)}\cos(\cos(\epsilon)\chi). \tag{110}$$

From (88) and (91) then we find by inputting

$$\phi = 0, \tag{111}$$

$$V_{P_S}^{\phi} = A\cos(\cos(\epsilon)(0)) + B\sin(\cos(\epsilon)(0)) \tag{112}$$

$$V_{P_S}^{\phi} = A \tag{113}$$

$$V_{P_S}^{\phi} = V_{P_1}^{\phi}\sin^2(\epsilon)\cos(\cos(\epsilon)\chi) + \frac{V_{P_1}^{\theta}}{\sin(\epsilon)}\sin(\cos(\epsilon)\chi), \tag{114}$$

and

$$V_{P_S}^{\theta} = \sin(\epsilon)\left(A\sin(\cos(\epsilon)(0)) - B\cos(\cos(\epsilon)(0))\right) \tag{115}$$

$$V_{P_s}^{\theta} = -B\sin(\epsilon) \tag{116}$$

$$V_{P_s}^{\theta} = V_{P_1}^{\theta}\cos(\cos(\epsilon)\chi) - V_{P_1}^{\phi}\sin^3(\epsilon)\sin(\cos(\epsilon)\chi). \tag{117}$$

We find the difference in the angle ξ between V_{P_1} and V_{P_s} by calculating the "dot product"

$$g(V_{P_1}, V_{P_s}) = \|V_{P_1}\|\|V_{P_s}\|\cos(\xi) \tag{118}$$

$$\cos(\xi) = \frac{g(V_{P_1}, V_{P_s})}{\|V_{P_1}\|\|V_{P_s}\|} \tag{119}$$

$$\cos(\xi) = \frac{g(V_{P_1}, V_{P_s})}{\sqrt{g(V_{P_1}, V_{P_1})}\sqrt{g(V_{P_s}, V_{P_s})}}. \tag{120}$$

We will also take the limit.

By the metric being diagonal,

$$g(V_{P_1}, V_{P_1}) = (V_{P_1}^{\theta})^2 g_{\theta\theta} + (V_{P_1}^{\phi})^2 g_{\phi\phi} \tag{121}$$

Taking the limit, we find

$$g(V_{P_1}, V_{P_1}) = (V_{P_1}^{\theta})^2(1) + \lim_{\epsilon\to 0}(V_{P_1}^{\phi})^2\sin^2\theta\Big|_{\epsilon=0} \tag{122}$$

$$g(V_{P_1}, V_{P_1}) = (V_{P_1}^{\theta})^2 + \lim_{\epsilon\to 0}(V_{P_1}^{\phi})^2\sin^2(\epsilon) \tag{123}$$

$$g(V_{P_1}, V_{P_1}) = (V_{P_1}^{\theta})^2. \tag{124}$$

$$\sqrt{g(V_{P_1}, V_{P_1})} = V_{P_1}^{\theta}. \tag{125}$$

$$g(V_{P_s}, V_{P_s}) = \lim_{\epsilon \to 0} \left(\begin{array}{l} \left(V_{P_1}^{\theta} \cos(\cos(\epsilon)\chi) - V_{P_1}^{\phi} \sin(\epsilon) \sin(\cos(\epsilon)\chi)\right)^2 \\ \sin^2(\epsilon) \left(\begin{array}{l} V_{P_1}^{\phi} \sin^2(\epsilon) \cos(\cos(\epsilon)\chi) \\ + \dfrac{V_{P_1}^{\theta}}{\sin(\epsilon)} \sin(\cos(\epsilon)\chi) \end{array} \right)^2 \end{array} \right) \tag{126}$$

This simplifies to

$$g(V_{P_s}, V_{P_s}) = \lim_{\epsilon \to 0} \left(\left(V_{P_1}^{\theta}\right)^2 \cos(\cos(\epsilon)\chi) + \left(V_{P_1}^{\theta}\right)^2 \sin(\cos(\epsilon)\chi) \right) \tag{127}$$

$$g(V_{P_s}, V_{P_s}) = \left(V_{P_1}^{\theta}\right)^2 \tag{128}$$

$$\sqrt{g(V_{P_s}, V_{P_s})} = V_{P_1}^{\theta}. \tag{129}$$

Now,

$$g(V_{P_1}, V_{P_s}) = \lim_{\epsilon \to 0} \left(\begin{array}{l} V_{P_1}^{\theta} \left(V_{P_1}^{\theta} \cos(\cos(\epsilon)\chi) - V_{P_1}^{\phi} \sin(\epsilon) \sin(\cos(\epsilon)\chi)\right) \\ V_{P_1}^{\phi} \sin^2(\epsilon) \left(\begin{array}{l} V_{P_1}^{\phi} \sin^2(\epsilon) \cos(\cos(\epsilon)\chi) \\ + \dfrac{V_{P_1}^{\theta}}{\sin(\epsilon)} \sin(\cos(\epsilon)\chi) \end{array} \right) \end{array} \right) \tag{130}$$

$$g(V_{P_1}, V_{P_s}) = \lim_{\epsilon \to 0} \left(\left(V_{P_1}^{\theta}\right)^2 \cos\left(\cos(\epsilon)\chi\right) + 0 \right) \tag{131}$$

$$g(V_{P_1}, V_{P_s}) = \left(V_{P_1}^{\theta}\right)^2 \cos(\chi) \,. \tag{132}$$

Plugging into (120) gives

$$\cos(\xi) = \frac{\left(V_{P_1}^{\theta}\right)^2 \cos(\chi)}{\left(V_{P_1}^{\theta}\right)\left(V_{P_1}^{\theta}\right)} \tag{133}$$

$$\cos(\xi) = \cos(\chi) \tag{134}$$

$$\boxed{\xi = \chi} \,. \tag{135}$$

Thus, the vector is rotated by the same angle it passed along the equatorial path.

3)

(3) Prove the Bianchi identity:

$$\nabla_{[a}R_{bc]de} = 0.$$

~~Recall that we used this identity in heuristically deriving the Einstein equation.~~

We base this off the May 19 lecture.

We can expand $\nabla_{[a}R_{bc]de}$ as a complete antisymmetrization

$$\nabla_{[a}R_{bc]de} = \frac{1}{6}\left[\nabla_a R_{bcde} - \nabla_a R_{cbde} + \nabla_b R_{cade} - \nabla_b R_{acde} + \nabla_c R_{abde} - \nabla_c R_{bade}\right]. \quad (1)$$

Considering the skew-symmetries

$$R_{abcd} = -R_{abdc} \quad (2)$$

and

$$R_{abcd} = -R_{badc}, \quad (3)$$

we find

$$\nabla_a R_{cbde} = -\nabla_a R_{bcde} \quad (4)$$

$$\nabla_b R_{acde} = -\nabla_b R_{cade} \quad (5)$$

$$\nabla_c R_{bade} = -\nabla_c R_{abde}. \quad (6)$$

So (1) simplifies to

$$\nabla_{[a}R_{bc]de} = \frac{1}{6}\left(2\nabla_a R_{bcde} + 2\nabla_b R_{cade} + 2\nabla_c R_{abde}\right) \quad (7)$$

$$\nabla_{[a}R_{bc]de} = \frac{1}{3}\left(\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde}\right). \quad (8)$$

Since we want to show that the LHS of (8) vanishes, we only need to show that the terms in the parentheses vanish.

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0 \tag{9}$$

Considering

$$\nabla g = 0, \tag{10}$$

we raise by g^{en}.

$$g^{en} \nabla_a R_{bcde} = \nabla_a (g^{en} R_{bcde}) . \tag{11}$$

This leads to

$$\nabla_a (g^{en} R_{bcde}) = \nabla_a R_{bcd}{}^{n} \tag{12}$$

Applying this contraction to the other two terms gives

$$g^{en} \nabla_b R_{cade} = \nabla_b R_{cad}{}^{n} \tag{13}$$

and

$$g^{en} \nabla_c R_{abde} = \nabla_c R_{abd}{}^{n} . \tag{14}$$

Thus, (9) becomes

$$\nabla_a R_{bcd}{}^{n} + \nabla_b R_{cad}{}^{n} + \nabla_c R_{abd}{}^{n} = 0 . \tag{15}$$

Therefore, in order to prove the Bianchi Identity, we just have to show that (15) holds.

We now follow the reasoning of Bohmer.

Consider that for a general rank (2,0) tensor T, with a Torsion-free connection

$$\nabla_a \nabla_b T_{cd} - \nabla_b \nabla_a T_{cd} = R_{abc}{}^{n} (T_{nd}) + R_{abd}{}^{n} (T_{cn}) . \tag{16}$$

Thus, for the covariant derivative of a one-form $\nabla_c \omega_d$, we get by (16)

$$\nabla_a \nabla_b \nabla_c \omega_d - \nabla_b \nabla_a \nabla_c \omega_d = R_{abc}{}^n (\nabla_n \omega_d) + R_{abd}{}^n (\nabla_c \omega_n). \quad (17)$$

For a one-form ω_d, we would get

$$\nabla_b \nabla_c \omega_d - \nabla_c \nabla_b \omega_d = -R^n{}_{dbc} \omega_n, \quad (18)$$

which by skew-symmetry is equivalent to

$$\nabla_b \nabla_c \omega_d - \nabla_c \nabla_b \omega_d = R_{bcd}{}^n \omega_n. \quad (19)$$

Applying ∇_a to (19), we get

$$\nabla_a \nabla_b \nabla_c \omega_d - \nabla_a \nabla_c \nabla_b \omega_d = \nabla_a (R_{bcd}{}^n \omega_n). \quad (20)$$

By the product rule, the RHS expands to give

$$\nabla_a \nabla_b \nabla_c \omega_d - \nabla_a \nabla_c \nabla_b \omega_d = (\nabla_a R_{bcd}{}^n) \omega_n + R_{bcd}{}^n (\nabla_a \omega_n). \quad (21)$$

We note that (17) and (21) should be equivalent expressions. We show this by considering the permutation of indices abc in (17).

$$\nabla_c \nabla_a \nabla_b \omega_d - \nabla_a \nabla_c \nabla_b \omega_d = R_{cab}{}^n (\nabla_n \omega_d) + R_{cad}{}^n (\nabla_b \omega_n) \quad (22)$$

$$\nabla_b \nabla_c \nabla_a \omega_d - \nabla_c \nabla_b \nabla_a \omega_d = R_{bca}{}^n (\nabla_n \omega_d) + R_{bcd}{}^n (\nabla_a \omega_n). \quad (23)$$

We now do the same for (21) to get

$$\nabla_c \nabla_a \nabla_b \omega_d - \nabla_c \nabla_b \nabla_a \omega_d = (\nabla_c R_{abd}{}^{n})\omega_n + R_{abd}{}^{n}(\nabla_c \omega_n) \quad (24)$$

$$\nabla_b \nabla_c \nabla_a \omega_d - \nabla_b \nabla_a \nabla_c \omega_d = (\nabla_b R_{cad}{}^{n})\omega_n + R_{cad}{}^{n}(\nabla_b \omega_n) . \quad (25)$$

Now, adding (17), (22), and (23), we get

$$\begin{aligned} &\nabla_a \nabla_b \nabla_c \omega_d - \nabla_b \nabla_a \nabla_c \omega_d && R_{abc}{}^{n}(\nabla_n \omega_d) + R_{abd}{}^{n}(\nabla_c \omega_n) \\ &+ \nabla_c \nabla_a \nabla_b \omega_d - \nabla_a \nabla_c \nabla_b \omega_d = && + R_{cab}{}^{n}(\nabla_n \omega_d) + R_{cad}{}^{n}(\nabla_b \omega_n) \quad (26) \\ &+ \nabla_b \nabla_c \nabla_a \omega_d - \nabla_c \nabla_b \nabla_a \omega_d && + R_{bca}{}^{n}(\nabla_n \omega_d) + R_{bcd}{}^{n}(\nabla_a \omega_n) . \end{aligned}$$

Considering the Bianchi identity that

$$R_{abc}{}^{n} + R_{bca}{}^{n} + R_{cab}{}^{n} = 0, \quad (27)$$

it follows that

$$R_{abc}{}^{n}(\nabla_n \omega_d) + R_{bca}{}^{n}(\nabla_n \omega_d) + R_{cab}{}^{n}(\nabla_n \omega_d) = 0 . \quad (28)$$

This means that (26) then simplifies to

$$\begin{aligned} &\nabla_a \nabla_b \nabla_c \omega_d - \nabla_b \nabla_a \nabla_c \omega_d && R_{abd}{}^{n}(\nabla_c \omega_n) + R_{cad}{}^{n}(\nabla_b \omega_n) \\ &+ \nabla_c \nabla_a \nabla_b \omega_d - \nabla_a \nabla_c \nabla_b \omega_d = && \quad (29) \\ &+ \nabla_b \nabla_c \nabla_a \omega_d - \nabla_c \nabla_b \nabla_a \omega_d && + R_{bcd}{}^{n}(\nabla_a \omega_n) . \end{aligned}$$

Now adding (21), (24), and (25), we get

$$
\begin{aligned}
&\nabla_a\nabla_b\nabla_c\omega_d - \nabla_a\nabla_c\nabla_b\omega_d && (\nabla_a R_{bcd}{}^{n})\omega_n + R_{bcd}{}^{n}(\nabla_a\omega_n) \\
&+\nabla_c\nabla_a\nabla_b\omega_d - \nabla_c\nabla_b\nabla_a\omega_d &=& +(\nabla_c R_{abd}{}^{n})\omega_n + R_{abd}{}^{n}(\nabla_c\omega_n) \\
&+\nabla_b\nabla_c\nabla_a\omega_d - \nabla_b\nabla_a\nabla_c\omega_d && +(\nabla_b R_{cad}{}^{n})\omega_n + R_{cad}{}^{n}(\nabla_b\omega_n).
\end{aligned}
\tag{30}
$$

It is easily seen that (29) and (30) have the same LHS, so the RHS are equal. We then cancel on both sides.

$$
\begin{aligned}
&\cancel{R_{abd}{}^{n}(\nabla_c\omega_n)} && (\nabla_a R_{bcd}{}^{n})\omega_n + \cancel{R_{bcd}{}^{n}(\nabla_a\omega_n)} \\
&+\cancel{R_{cad}{}^{n}(\nabla_b\omega_n)} &=& +(\nabla_c R_{abd}{}^{n})\omega_n + \cancel{R_{abd}{}^{n}(\nabla_c\omega_n)} \\
&+\cancel{R_{bcd}{}^{n}(\nabla_a\omega_n)} && +(\nabla_b R_{cad}{}^{n})\omega_n + \cancel{R_{cad}{}^{n}(\nabla_b\omega_n)}.
\end{aligned}
\tag{31}
$$

This yields

$$(\nabla_a R_{bcd}{}^{n})\omega_n + (\nabla_b R_{cad}{}^{n})\omega_n + (\nabla_c R_{abd}{}^{n})\omega_n = 0 \tag{32}$$

$$(\nabla_a R_{bcd}{}^{n} + \nabla_b R_{cad}{}^{n} + \nabla_c R_{abd}{}^{n})\,\omega_n = 0\,. \tag{33}$$

Since ω_n represents any one-form, then in general

$$\nabla_a R_{bcd}{}^{n} + \nabla_b R_{cad}{}^{n} + \nabla_c R_{abd}{}^{n} = 0\,. \tag{34}$$

Thus, we have shown (15) and have consequently proven

$$\boxed{\nabla_{[a}R_{bc]de} = 0}\,. \tag{35}$$

4)

(4) Consider freely-falling masses in the "warped time" spacetime

$$g = -(1 + 2\phi(\mathrm{x}))\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2,$$

where $|\phi(\mathrm{x})| \ll 1$. Compute all the components of the Riemann tensor up to first-order in ϕ. Write out the geodesic deviation equation in the coordinate system (t, x, y, z) and compare this to the relative acceleration between freely-falling masses in Newtonian gravity.

We have the diagonal metric

$$g = -(1+2\phi(\vec{x}))dt^2 + dx^2 + dy^2 + dz^2, \qquad (1)$$

where

$$|\phi(\vec{x})| \ll 1. \qquad (2)$$

We can calculate the Christoffel Symbols from the Euler-Lagrange equations we use for geodesics.

Consider that in general, the geodesic equations take the form

$$\ddot{x}^\sigma + \Gamma^\sigma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0. \qquad (3)$$

We take the Lagrangian as

$$L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta. \qquad (4)$$

It can be seen that the metric in (1) is diagonal, so we expect two geodesic equations.

Reading off the metric, (4) gives

$$L = -(1+2\phi(\vec{x}))\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \qquad (5)$$

Where the $\cdot$ represents a derivative wrt. our parameter τ.

For x^j we have have the Euler-Lagrange equations

$$\frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{x}^j}\right] = \frac{\partial L}{\partial x^j} \tag{6}$$

$$\frac{d}{d\tau}(2\dot{x}^j) = -\partial_j\left(1+2\phi(\vec{x})\right)\dot{t}^2 \tag{7}$$

$$2\ddot{x}^j = -2\,\partial_j\phi(\vec{x})\,\dot{t}^2 \tag{8}$$

$$\ddot{x}^j = -\partial_j\phi(\vec{x})\,\dot{t}^2. \tag{9}$$

We put (9) in the form of (3) and switching index to i

$$\ddot{x}^i + \partial_i\,\phi(\vec{x})\,\dot{t}^2 = 0. \tag{10}$$

For the time component, we have

$$\frac{d}{d\tau}\left(-\left(1+2\phi(\vec{x})\right)2\dot{t}\right) = \partial_t\left(-\left(1+2\phi(\vec{x})\right)\dot{t}^2\right) \tag{11}$$

$$-2\ddot{t}\left(1+2\phi(\vec{x})\right) - 2\dot{t}\left(2\left(\partial_t\phi(\vec{x})\dot{t} + \partial_i\phi(\vec{x})\dot{x}^i\right)\right) = -2\partial_t\phi(\vec{x})\,\dot{t}^2 \tag{12}$$

$$\ddot{t}\left(1+2\phi(\vec{x})\right) + \partial_t\phi(\vec{x})\dot{t}^2 + 2\partial_i\phi(\vec{x})\dot{t}\dot{x}^i = 0 \tag{13}$$

Putting this in the form of (3) gives

$$\ddot{t} + \frac{\partial_t\phi(\vec{x})}{1+2\phi(\vec{x})}\dot{t}^2 + \frac{2\partial_i\phi(\vec{x})}{1+2\phi(\vec{x})}\dot{t}\dot{x}^i = 0. \tag{14}$$

Reading off (10) then

$$\Gamma^{i}_{tt} = \partial_i \phi(\vec{x}) . \tag{15}$$

Reading off (14) then

$$\Gamma^{t}_{tt} = \frac{\partial_t \phi(\vec{x})}{1+2\phi(\vec{x})} , \tag{16}$$

and

$$\Gamma^{t}_{it} = \frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})} \tag{17}$$

$$\Gamma^{t}_{ti} = \frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})} . \tag{18}$$

With all other Christoffels going to zero since for example, we never get terms with $\dot{x}^{i^2}$, and, $\dot{x}^i \dot{x}^j$

The geodesic deviation equation is

$$\frac{D^2 z^a}{d\tau^2} = -R^a{}_{dcb}\, u^d z^c u^b . \tag{19}$$

The Riemann Tensor is calculated as

$$R^d{}_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac} \Gamma^d_{be} - \Gamma^e_{ab} \Gamma^d_{ce} . \tag{20}$$

Let's start calculating the Riemann Tensor components.

We keep in mind the identities that

$$R^d{}_{abc} = -R^d{}_{acb} \tag{21}$$

$$R^d{}_{abc} + R^d{}_{bca} + R^d{}_{cab} = 0. \tag{22}$$

Automatically by (21),

$$R^d{}_{aii} = 0 \tag{23}$$

$$R^d{}_{att} = 0. \tag{24}$$

Now,

$$R^i{}_{abc} = \partial_b \Gamma^i_{ac} - \partial_c \Gamma^i_{ab} + \Gamma^e_{ac}\Gamma^i_{be} - \Gamma^e_{ab}\Gamma^i_{ce}. \tag{25}$$

From (15) and (21), we note that only $R^i{}_{ttc}, R^i{}_{tbt}$ can be non-zero for those with up index i.

$$R^i{}_{ttj} = \partial_t \Gamma^i_{tj} - \partial_j \Gamma^i_{tt} + \Gamma^e_{tj}\Gamma^i_{te} - \Gamma^e_{tt}\Gamma^i_{je} \tag{26}$$

$$R^i{}_{ttj} = \partial_t(0) - \partial_j(\partial_i\phi) + \Gamma^t_{tj}\Gamma^i_{tt} - \left(\Gamma^e_{tt}(0)\right) \tag{27}$$

$$R^i{}_{ttj} = -\partial_j\partial_i\phi + \frac{\partial_j\phi\,\partial_t\phi}{(1+2\phi)^2}. \tag{28}$$

By the skew-symmetry in (21),

$$R^{i}{}_{tjt} = \partial_j \partial_i \phi - \frac{\partial_j \phi \, \partial_t \phi}{(1+2\phi)^2} . \tag{29}$$

For up index t, we have $R^t{}_{iit}$, $R^t{}_{iti}$, $R^t{}_{tti}$, $R^t{}_{tit}$, $R^t{}_{ijt}$, and $R^t{}_{itj}$.

$$R^t{}_{abc} = \partial_b \Gamma^t_{ac} - \partial_c \Gamma^t_{ab} + \Gamma^e_{ac} \Gamma^t_{be} - \Gamma^e_{ab} \Gamma^t_{ce} \tag{30}$$

For $R^t{}_{iit}$,

$$R^t{}_{iit} = \partial_i \Gamma^t_{it} - \partial_t \Gamma^t_{ii} + \Gamma^e_{it} \Gamma^t_{ie} - \Gamma^e_{ii} \Gamma^t_{te} \tag{31}$$

$$R^t{}_{iit} = \partial_i \left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})} \right) - \partial_t(0) + \Gamma^t_{it} \Gamma^t_{it} - (0) \Gamma^t_{te} \tag{32}$$

$$R^t{}_{iit} = \partial_i \left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})} \right) + \frac{(\partial_i \phi(\vec{x}))^2}{(1+2\phi(\vec{x}))^2} . \tag{33}$$

By skew-symmetry in (21),

$$R^t{}_{iti} = -\partial_i \left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})} \right) - \frac{(\partial_i \phi(\vec{x}))^2}{(1+2\phi(\vec{x}))^2} . \tag{34}$$

Now, for $R^t{}_{tti}$

$$R^t{}_{tti} = \partial_t \Gamma^t_{ti} - \partial_i \Gamma^t_{tt} + \Gamma^e_{ti} \Gamma^t_{te} - \Gamma^e_{tt} \Gamma^t_{te} \tag{35}$$

$$R^{t}{}_{tti} = \begin{gathered} \partial_t\left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})}\right) - \partial_i\left(\frac{\partial_t \phi(\vec{x})}{1+2\phi(\vec{x})}\right) + \Gamma^{t}_{ti}\Gamma^{t}_{tt} \\ -\left(\Gamma^{i}_{tt}\Gamma^{t}_{it} + \Gamma^{t}_{tt}\Gamma^{t}_{tt}\right) \end{gathered} \tag{36}$$

$$R^{t}{}_{tti} = \begin{gathered} \partial_t\left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})}\right) - \partial_i\left(\frac{\partial_t \phi(\vec{x})}{1+2\phi(\vec{x})}\right) + \\ \frac{\partial_i\phi(\vec{x})\,\partial_t\phi(\vec{x})}{(1+2\phi(\vec{x}))^2} - \frac{(\partial_i\phi(\vec{x}))^2}{1+2\phi(\vec{x})} - \frac{(\partial_t\phi(\vec{x}))^2}{(1+2\phi(\vec{x}))^2} \end{gathered} . \tag{37}$$

By skew-symmetry in (21),

$$R^{t}{}_{tit} = \begin{gathered} -\partial_t\left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})}\right) + \partial_i\left(\frac{\partial_t \phi(\vec{x})}{1+2\phi(\vec{x})}\right) \\ -\frac{\partial_i\phi(\vec{x})\,\partial_t\phi(\vec{x})}{(1+2\phi(\vec{x}))^2} + \frac{(\partial_i\phi(\vec{x}))^2}{1+2\phi(\vec{x})} + \frac{(\partial_t\phi(\vec{x}))^2}{(1+2\phi(\vec{x}))^2} \end{gathered} . \tag{38}$$

Now,

$$R^{t}{}_{ijt} = \partial_j \Gamma^{t}_{it} - \partial_t \Gamma^{t}_{ij} + \Gamma^{e}_{it}\Gamma^{t}_{je} - \Gamma^{e}_{ij}\Gamma^{t}_{te} \tag{39}$$

$$R^{t}{}_{ijt} = \partial_j\left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})}\right) - \partial_t(0) + \Gamma^{t}_{it}\Gamma^{t}_{jt} - (0)\Gamma^{t}_{te} \tag{40}$$

$$R^{t}{}_{ijt} = \partial_j\left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})}\right) + \frac{\partial_i\phi(\vec{x})\,\partial_j\phi(\vec{x})}{(1+2\phi(\vec{x}))^2} . \tag{41}$$

By skew-symmetry in (21),

$$R^t{}_{itj} = -\partial_j\left(\frac{\partial_i \phi(\vec{x})}{1+2\phi(\vec{x})}\right) - \frac{\partial_i \phi(\vec{x})\,\partial_j \phi(\vec{x})}{(1+2\phi(\vec{x}))^2} \quad (42)$$

However, if we take only 1st-Order terms,

$$R^i{}_{ttj} = -\partial_j \partial_i \phi(\vec{x}) \quad (43)$$

$$R^i{}_{tjt} = \partial_j \partial_i \phi(\vec{x}) \quad (44)$$

$$R^t{}_{iit} = 0 \quad (45)$$

$$R^t{}_{iti} = 0 \quad (46)$$

$$R^t{}_{tti} = 0 \quad (47)$$

$$R^t{}_{tit} = 0 \quad (48)$$

$$R^t{}_{ijt} = 0 \quad (49)$$

$$R^t{}_{itj} = 0. \quad (50)$$

From (19), using (44),

$$\frac{D^2 z^i}{d\tau^2} = -R^i{}_{tjt} u^t z^j u^t \quad (51)$$

$$\frac{D^2 z^i}{d\tau^2} = -(\partial_i \partial_j \phi(\vec{x})) u^t z^j u^t. \quad (52)$$

In comoving coordinates

$$u^\alpha \overset{*}{=} (1,0,0,0). \tag{52}$$

Therefore, (52) can be written as

$$\frac{D^2 z^i}{d\tau^2} = -(\partial_i \partial_j \phi(\vec{x}))(1) z^j(1) \tag{53}$$

$$\frac{D^2 z^i}{d\tau^2} = -(\partial_i \partial_j \phi(\vec{x})) z^j. \tag{54}$$

We note that in Newtonian gravity, acceleration is given as

$$a^i = -g^{ij} \partial_i \phi(\vec{x}). \tag{55}$$

$$\vec{a} = -\vec{\nabla}\phi(\vec{x}) \tag{56}$$

Note that if the metric is flat

$$\partial_i = \partial^i, \tag{57}$$

which this is.

Thus,

$$\frac{D^2 z^i}{d\tau^2} = -(\partial^i \partial_j \phi(\vec{x})) z^j. \tag{58}$$

We may then interpret this as

$$-(\partial^i \partial_j \phi(\vec{x})) = -\nabla^2 \phi(\vec{x}). \tag{59}$$

In terms of the acceleration in (56), this is

$$\boxed{-\nabla^2\phi(\vec{x}) = \vec{\nabla}\cdot\vec{a}\,.} \tag{60}$$

Thus, $\left(-\partial^i\partial_j\,\phi(\vec{x})\right)$ can be thought of

as relative acceleration.

The geodesic deviation can be thought of as the relative acceleration of geodesics, as z^i represents the geodesic deviation vector.

While we demonstrated this as an analog to Newtonian gravity in comoving coordinates,

$$\boxed{R^i{}_{tjt} = \partial^i\partial_j\,\phi(\vec{x})} \tag{61}$$

this suggests a more general relation of

$$\boxed{R^i{}_{ajb}\,u^a u^b = \partial^i\partial_j\,\phi(\vec{x})\,.} \tag{62}$$

5)

(5) Riemann curvature in lower dimensions. For a two-dimensional manifold, show that $R_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R$. The Riemann curvature can thus be expressed algebraically in terms of the metric and the scalar curvature. As an explicit example, calculate left- and right-hand sides of this equation for a sphere using the usual coordinates of Problem #2.

We consider that by construction, the Ricci tensor is given by

$$R_{ac} = g^{bd} R_{abcd}, \qquad (1)$$

and the Ricci scalar as

$$R = g^{ac} R_{ac}. \qquad (2)$$

Thus,

$$R = g^{ac} g^{bd} R_{abcd}. \qquad (3)$$

We note that the Riemann curvature has the anti-symmetry

$$R_{abcd} = -R_{abdc}, \qquad (4)$$

and the identity

$$R_{abcd} + R_{acbd} + R_{adbc} = 0. \qquad (5)$$

Therefore,

$$R_{[abcd]} = 0 \qquad (6)$$

We showed in class that these symmetries imply that R_{abcd} has

$$\# = \frac{1}{12} D^2 (D^2 - 1). \qquad (7)$$

independent components.

Thus for two dimensions, we have

$$\#\big|_{D=2} = 1 \qquad (8)$$

independent component.

This will help us later, when we use coordinates.

This also then hints that in 2D,

$$R_{abcd} \propto R. \tag{9}$$

We note that in 2D

$$g_{ac}g_{bd} - g_{ad}g_{bc} = g_{ac}g_{bd} - g_{ad}g_{cb}. \tag{10}$$

From (3)

$$g_{ac}g_{bd}R = g_{ac}g_{bd}g^{ac}g^{bd}R_{abcd} \tag{11}$$

$$g_{ac}g_{bd}R = R_{abcd}. \tag{12}$$

Similarly,

$$g_{ad}g_{cb}R = g_{ad}g_{cb}g^{ad}g^{cb}R_{acdb} \tag{13}$$

$$g_{ad}g_{cb}R = R_{acdb}. \tag{14}$$

Since we showed that there is only one independent component in 2D, we can denote it with symmetries

$$R_{1212} = R_{2121} \tag{15}$$

$$R_{1212} = -R_{2112} \tag{16}$$

$$R_{1212} = -R_{1221}. \tag{17}$$

This is also shown by the fact that (4) in 2D makes

$$R_{11cd} = 0 \tag{18}$$

$$R_{22cd} = 0 \tag{19}$$

$$R_{ab11} = 0 \tag{20}$$

$$R_{ab22} = 0 . \tag{21}$$

For example

$$R_{1111} = -R_{1111} , \tag{22}$$

So

$$R_{1111} = 0 , \tag{23}$$

as we "switched" these two 1 indices.

Thus,

$$R_{acdb} = -R_{abcd} . \tag{24}$$

Therefore,

$$R_{abcd} - R_{acdb} = 2R_{abcd} \tag{25}$$

$$g_{ac}g_{bd}R - g_{ad}g_{cb}R = 2R_{abcd} . \tag{26}$$

Factoring, and considering (10)

$$2R_{abcd} = (g_{ac}g_{bd} - g_{ad}g_{bc})R \tag{27}$$

$$\boxed{R_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R} . \tag{28}$$

From Problem 2, we had the metric

$$g = d\theta^2 + \sin^2\theta \, d\phi^2, \tag{29}$$

and the only non-zero Christoffels as

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \tag{30}$$

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta \tag{31}$$

$$\Gamma^{\phi}_{\phi\theta} = \cot\theta . \tag{32}$$

We note that by definition

$$R^{a}{}_{bcd} = \partial_c \Gamma^{a}_{bd} - \partial_d \Gamma^{a}_{bc} + \Gamma^{a}_{ce}\Gamma^{e}_{bd} - \Gamma^{a}_{de}\Gamma^{e}_{bc} . \tag{33}$$

We showed previously, that we in fact only have to calculate

$$R^{\theta}{}_{\phi\theta\phi} = \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{\theta e}\Gamma^{e}_{\phi\phi} - \Gamma^{\theta}_{\phi e}\Gamma^{e}_{\phi\theta} \tag{34}$$

From (30), (31), and (32), we note that most of these terms are 0, so it simplifies to

$$R^{\theta}{}_{\phi\theta\phi} = \partial_\theta \Gamma^{\theta}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} \tag{35}$$

$$R^{\theta}{}_{\phi\theta\phi} = \frac{\partial}{\partial\theta}\left(-\sin\theta\cos\theta\right) - \left(-\sin\theta\cos\theta\right)\cot\theta \tag{36}$$

$$R^{\theta}{}_{\phi\theta\phi} = \left(-\cos^2\theta + \sin^2\theta\right) - \left(-\cos^2\theta\right) \tag{37}$$

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \, . \tag{38}$$

Contracting this, we get, considering the metric being diagonal

$$g^{\theta e} R^{\theta}{}_{\phi\theta\phi} = g^{\theta\theta} R^{\theta}{}_{\phi\theta\phi} + g^{\theta\phi} R^{\theta}{}_{\phi\theta\phi} \tag{39}$$

$$g^{\theta e} R^{\theta}{}_{\phi\theta\phi} = g^{\theta\theta} R^{\theta}{}_{\phi\theta\phi} + 0 \tag{40}$$

$$R_{\theta\phi\theta\phi} = (1)(\sin^2\theta) \tag{41}$$

$$R_{\theta\phi\theta\phi} = \sin^2\theta \, . \tag{42}$$

This serves as the LHS of the equation.

Now, we deal with the RHS.

We note that since the metric is diagonal,

$$g_{\theta\phi}\, g_{\phi\theta} = 0 \, . \tag{43}$$

Thus,

$$\frac{1}{2}(g_{\theta\theta}\, g_{\phi\phi} - g_{\theta\phi}\, g_{\phi\theta})R = \frac{1}{2} g_{\theta\theta}\, g_{\phi\phi}\, R \tag{44}$$

$$\frac{1}{2}(g_{\theta\theta}\, g_{\phi\phi} - g_{\theta\phi}\, g_{\phi\theta})R = \frac{1}{2} \sin^2\theta \, R \, . \tag{45}$$

To get the Ricci scalar, we first calculate the Ricci Tensor via (1).

So

$$R_{\theta\theta} = g^{bd} R_{\theta b\theta d}, \tag{46}$$

which by the metric being diagonal simplifies to

$$R_{\theta\theta} = g^{\theta\theta} R_{\theta\theta\theta\theta} + g^{\phi\phi} R_{\theta\phi\theta\phi}. \tag{47}$$

$$R_{\theta\theta} = g^{\phi\phi} R_{\theta\phi\theta\phi} \tag{48}$$

$$R_{\theta\theta} = \frac{1}{\sin^2\theta} \sin^2\theta \tag{49}$$

$$R_{\theta\theta} = 1 \tag{50}$$

Similarly,

$$R^{\phi\phi} = g^{bd} R_{\phi b\phi d} \tag{51}$$

$$R^{\phi\phi} = g^{\theta\theta} R_{\phi\theta\phi\theta} + g^{\phi\phi} R_{\phi\phi\phi\phi} \tag{52}$$

$$R^{\phi\phi} = g^{\theta\theta} R_{\phi\theta\phi\theta}. \tag{53}$$

$$R^{\phi\phi} = \sin^2\theta \tag{54}$$

By (2) and the metric being diagonal, we don't care about $R_{\theta\phi}$ and $R_{\phi\theta}$. Therefore,

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} \tag{55}$$

$$R = (1)(1) + \frac{1}{\sin^2\theta} \sin^2\theta \tag{56}$$

$$R = 2. \tag{57}$$

Plugging this into (45) gives

$$\frac{1}{2}(g_{\theta\theta}\, g_{\phi\phi} - g_{\theta\phi}\, g_{\phi\theta})R = \frac{1}{2}\sin^2\theta\,(2) \tag{58}$$

$$\boxed{\frac{1}{2}(g_{\theta\theta}\, g_{\phi\phi} - g_{\theta\phi}\, g_{\phi\theta})R = \sin^2\theta} \; . \tag{59}$$

We see that (59), the RHS, exactly matches (42), the LHS.

By (15), (16), and (17) we also get

$$R_{\phi\theta\phi\theta} = \sin^2\theta \, , \tag{60}$$

$$R_{\phi\theta\theta\phi} = -\sin^2\theta \, , \tag{61}$$

and

$$R_{\theta\phi\phi\theta} = -\sin^2\theta \, . \tag{62}$$

6)

~~(6) Show that $\partial_{[\mu}F_{\nu\lambda]} = 0$ contains both $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ as its components. Explain with an explicit calculation how one can get the Poynting vector from the electromagnetic stress-energy tensor.~~

Consider that the Faraday tensor is given by

$$F^{\nu\lambda} = \begin{pmatrix} 0 & \frac{E^x}{c} & \frac{E^y}{c} & \frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B^z & -B^y \\ -\frac{E^y}{c} & -B^z & 0 & B^x \\ -\frac{E^z}{c} & B^y & -B^x & 0 \end{pmatrix} . \tag{1}$$

Considering the Lorentzian metric

$$\eta_{\nu\alpha} = \mathrm{diag}(-1,1,1,1) , \tag{2}$$

we get the lowered index version of (1) as

$$F_{\nu\lambda} = \eta_{\beta\lambda} F^{\alpha\beta} \eta_{\nu\alpha} \tag{3}$$

$$F_{\nu\lambda} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{E^x}{c} & \frac{E^y}{c} & \frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B^z & -B^y \\ -\frac{E^y}{c} & -B^z & 0 & B^x \\ -\frac{E^z}{c} & B^y & -B^x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

$$F_{\nu\lambda} = \begin{pmatrix} 0 & -\frac{E^x}{c} & -\frac{E^y}{c} & -\frac{E^z}{c} \\ \frac{E^x}{c} & 0 & B^z & -B^y \\ \frac{E^y}{c} & -B^z & 0 & B^x \\ \frac{E^z}{c} & B^y & -B^x & 0 \end{pmatrix} . \tag{5}$$

We note that

$$\partial_{[\mu}F_{\nu\lambda]} = \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} - \partial_\mu F_{\lambda\nu} - \partial_\nu F_{\mu\lambda} - \partial_\lambda F_{\nu\mu} . \tag{6}$$

Using anti-symmetry, this becomes

$$\partial_{[\mu}F_{\nu\lambda]} = \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\lambda} \quad (7)$$

$$\partial_{[\mu}F_{\nu\lambda]} = 2\left(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}\right). \quad (8)$$

Considering the condition that

$$\partial_{[\mu}F_{\nu\lambda]} = 0, \quad (9)$$

Suggests that

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \quad (10)$$

We consider the convention that

$$i = 0,1,2,3 \leftrightarrow ct, x, y, z. \quad (11)$$

By inspection of (5), and taking

$$\lambda = 1, \mu = 2, \nu = 3, \quad (12)$$

we see that

$$\boxed{\partial_2 F_{31} + \partial_3 F_{12} + \partial_1 F_{23} = 0} \quad (13)$$

is equal to

$$\boxed{\frac{\partial B^y}{\partial y} + \frac{\partial B^z}{\partial z} + \frac{\partial B^x}{\partial x} = 0}. \quad (14)$$

This is of course equivalent to

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}. \quad (15)$$

We note that for an arbitrary vector $\vec{V}$, the curl is given by

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial V^z}{\partial y} - \frac{\partial V^y}{\partial z}\right)\hat{x} + \left(\frac{\partial V^x}{\partial z} - \frac{\partial V^z}{\partial x}\right)\hat{y} + \left(\frac{\partial V^y}{\partial x} - \frac{\partial V^x}{\partial y}\right)\hat{z} \,. \qquad (16)$$

Thus, we can make use of this to find

$$\vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0 \,. \qquad (17)$$

By inspection of (5), and taking

$$\lambda = 0,\ \mu = 2,\ \nu = 3, \qquad (18)$$

we get

$$\partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = 0 \,. \qquad (19)$$

This is equal to

$$\partial_2 \frac{E^z}{c} - \partial_3 \frac{E^y}{c} + \partial_0 B^x = 0 \qquad (20)$$

$$\frac{1}{c}\left(\frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z}\right) + \frac{1}{c}\frac{\partial B^x}{\partial t} = 0 \qquad (21)$$

Multiplying both sides of (21) by c, we get

$$\left(\frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z}\right) + \frac{\partial B^x}{\partial t} = 0 \,. \qquad (22)$$

This can be taken as the x-component of (17)

$$\boxed{\left(\vec{\nabla}\times\vec{E} + \partial_t\vec{B}\right)^x = \left(\frac{\partial E^z}{\partial y} - \frac{\partial E^y}{\partial z}\right) + \frac{\partial B^x}{\partial t} = 0} \,. \qquad (23)$$

Similarly we find that taking

$$\lambda = 0, \mu = 1, \nu = 3, \tag{24}$$

we get

$$\partial_1 F_{30} + \partial_3 F_{01} + \partial_0 F_{13} = 0 . \tag{25}$$

This is equal to

$$\partial_1 \frac{E^z}{c} - \partial_3 \frac{E^x}{c} - \partial_0 B^y = 0 . \tag{26}$$

Multiplying both sides by -1 yields

$$\frac{1}{c}\left(\frac{\partial E^x}{\partial z} - \frac{\partial E^z}{\partial x}\right) + \frac{1}{c}\frac{\partial B^y}{\partial t} = 0 . \tag{27}$$

As for the x-component, we get for the y-component

$$\boxed{\left(\vec{\nabla} \times \vec{E} + \partial_t \vec{B}\right)^y = \left(\frac{\partial E^x}{\partial z} - \frac{\partial E^z}{\partial x}\right) + \frac{\partial B^y}{\partial t} = 0} . \tag{28}$$

Similarly we find that taking

$$\lambda = 0, \mu = 1, \nu = 2, \tag{29}$$

we get

$$\partial_1 F_{20} + \partial_2 F_{01} + \partial_0 F_{12} = 0 . \tag{30}$$

This is equal to

$$\partial_1 \frac{E^y}{c} - \partial_2 \frac{E^x}{c} + \partial_0 B^z = 0 . \tag{31}$$

This yields

$$\frac{1}{c}\left(\frac{\partial E^y}{\partial x} - \frac{\partial E^x}{\partial y}\right) + \frac{1}{c}\frac{\partial B^z}{\partial t} = 0 . \tag{32}$$

$$\boxed{\left(\vec{\nabla} \times \vec{E} + \partial_t \vec{B}\right)^z = \left(\frac{\partial E^y}{\partial x} - \frac{\partial E^x}{\partial y}\right) + \frac{1}{c}\frac{\partial B^z}{\partial t} = 0} . \tag{33}$$

Thus, we indeed recover (17) from the components of $\partial_{[\mu}F_{\nu\lambda]}$.

Griffiths defines the Poynting vector as

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right). \tag{33}$$

$$\vec{S} = \frac{1}{\mu_0}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E^x & E^y & E^z \\ B^x & B^y & B^z \end{vmatrix} \tag{34}$$

$$\vec{S} = \frac{1}{\mu_0}\left(\left(E^yB^z - E^zB^y\right)\hat{x} + \left(E^zB^x - E^xB^z\right)\hat{y} + \left(E^xB^y - E^yB^x\right)\hat{z}\right). \tag{35}$$

The Electromagnetic Stress-Energy Tensor is given by

$$T^{\mu\nu}_{(em)} = \frac{1}{\mu_0}\left(F^{\lambda\mu}F_{\lambda}{}^{\nu} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right). \tag{36}$$

We calculate $F_{\mu}{}^{\nu}$ as

$$F_{\mu}{}^{\nu} = \eta_{\mu\gamma}F^{\mu\nu} \tag{37}$$

$$F_{\mu}{}^{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \frac{E^x}{c} & \frac{E^y}{c} & \frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B^z & -B^y \\ -\frac{E^y}{c} & -B^z & 0 & B^x \\ -\frac{E^z}{c} & B^y & -B^x & 0 \end{pmatrix} \tag{38}$$

$$F_\mu{}^\nu = \begin{pmatrix} 0 & -\frac{E^x}{c} & -\frac{E^y}{c} & -\frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B^z & -B^y \\ -\frac{E^y}{c} & -B^z & 0 & B^x \\ -\frac{E^z}{c} & B^y & -B^x & 0 \end{pmatrix} . \quad (39)$$

The Poynting vector is interpreted as an "energy flux" along the direction or as the "momentum density" of the i-th component. Thus, we only need concern ourselves with the components $T^{0j}_{(em)}$ and $T^{i0}_{(em)}$ respectively.

From (36) then, the second term vanishes for these terms, since $\eta^{\mu\nu}$ is diagonal

$$T^{0j}_{(em)} = \frac{1}{\mu_0} F^{\lambda 0} F_\lambda{}^j \quad (39)$$

$$T^{i0}_{(em)} = \frac{1}{\mu_0} F^{\lambda i} F_\lambda{}^0 . \quad (40)$$

Starting with (39),

$$T^{01}_{(em)} = \frac{1}{\mu_0} F^{\lambda 0} F_\lambda{}^1 \quad (41)$$

$$T^{01}_{(em)} = \frac{1}{\mu_0}\left((0)\left(\frac{E^x}{c}\right) + \left(\frac{-E^x}{c}\right)(0) + \left(\frac{-E^y}{c}\right)(-B^z) + \left(\frac{-E^z}{c}\right)(B^y)\right) \quad (42)$$

$$T^{01}_{(em)} = \frac{1}{\mu_0}\left(\frac{E^y B^z}{c} - \frac{E^z B^y}{c}\right) . \quad (43)$$

From (35), this is clearly

$$\boxed{T^{01}_{(em)} = \frac{S^x}{c}} . \quad (44)$$

$$T^{02}_{(em)} = \frac{1}{\mu_0} F^{\lambda 0} F_{\lambda}{}^{2} \tag{45}$$

$$T^{02}_{(em)} = \frac{1}{\mu_0}\left((0)\left(\frac{E^y}{c}\right) + \left(\frac{-E^x}{c}\right)(B^z) + \left(\frac{-E^y}{c}\right)(0) + \left(\frac{-E^z}{c}\right)(-B^x)\right) \tag{46}$$

$$T^{02}_{(em)} = \frac{1}{\mu_0}\left(\frac{E^z B^x}{c} - \frac{E^x B^z}{c}\right) \tag{47}$$

$$\boxed{T^{02}_{(em)} = \frac{S^y}{c}}\,. \tag{48}$$

$$T^{03}_{(em)} = \frac{1}{\mu_0} F^{\lambda 0} F_{\lambda}{}^{3} \tag{49}$$

$$T^{03}_{(em)} = \frac{1}{\mu_0}\left((0)\left(\frac{E^z}{c}\right) + \left(\frac{-E^x}{c}\right)(-B^y) + \left(\frac{-E^y}{c}\right)(B^x) + \left(\frac{-E^z}{c}\right)(0)\right) \tag{50}$$

$$T^{03}_{(em)} = \frac{1}{\mu_0}\left(\frac{E^x B^y}{c} - \frac{E^y B^x}{c}\right) \tag{51}$$

$$\boxed{T^{03}_{(em)} = \frac{S^z}{c}}\,. \tag{52}$$

Now, dealing with (40), we find

$$T^{10}_{(em)} = \frac{1}{\mu_0} F^{\lambda 1} F_{\lambda}{}^{0} \tag{53}$$

$$T^{10}_{(em)} = \frac{1}{\mu_0}\left(\left(\frac{E^x}{c}\right)(0) + (0)\left(\frac{-E^x}{c}\right) + (-B^z)\left(\frac{-E^y}{c}\right) + (B^y)\left(\frac{-E^z}{c}\right)\right) \tag{54}$$

$$T^{10}_{(em)} = \frac{1}{\mu_0}\left(\frac{E^y B^z}{c} - \frac{E^z B^y}{c}\right) \tag{55}$$

$$\boxed{T^{10}_{(em)} = \frac{S^x}{c}}\,. \tag{56}$$

$$T^{20}_{(em)} = \frac{1}{\mu_0} F^{\lambda 2} F_{\lambda}{}^{0} \tag{57}$$

$$T^{20}_{(em)} = \frac{1}{\mu_0}\left(\left(\frac{E^y}{c}\right)(0) + (B^z)\left(\frac{-E^x}{c}\right) + (0)\left(\frac{-E^y}{c}\right) + (-B^x)\left(\frac{-E^z}{c}\right)\right) \tag{58}$$

$$T^{20}_{(em)} = \frac{1}{\mu_0}\left(\frac{E^z B^x}{c} - \frac{E^x B^z}{c}\right) \tag{59}$$

$$\boxed{T^{20}_{(em)} = \frac{S^y}{c}}. \tag{60}$$

$$T^{30}_{(em)} = \frac{1}{\mu_0} F^{\lambda 3} F_{\lambda}{}^{0} \tag{61}$$

$$T^{20}_{(em)} = \frac{1}{\mu_0}\left(\left(\frac{E^z}{c}\right)(0) + (-B^y)\left(\frac{-E^x}{c}\right) + (B^x)\left(\frac{-E^y}{c}\right) + (0)\left(\frac{-E^z}{c}\right)\right) \tag{62}$$

$$T^{30}_{(em)} = \frac{1}{\mu_0}\left(\frac{E^x B^y}{c} - \frac{E^y B^x}{c}\right) \tag{63}$$

$$\boxed{T^{30}_{(em)} = \frac{S^z}{c}}. \tag{64}$$

Therefore,

$$T^{\mu\nu}_{(em)} = \begin{pmatrix} \cdot & \frac{S^x}{c} & \frac{S^y}{c} & \frac{S^z}{c} \\ \frac{S^x}{c} & \cdot & \cdot & \cdot \\ \frac{S^y}{c} & \cdot & \cdot & \cdot \\ \frac{S^z}{c} & \cdot & \cdot & \cdot \end{pmatrix}. \tag{65}$$

We have thus retrieved the components of the Poynting Vector, scaled by $\frac{1}{c}$, as components of the Electromagnetic Stress-Energy tensor.

7) (7) In flat spacetime, write out the time- and spatial- components of the conservation law, $\nabla_a T^{ab} = 0$, for a perfect fluid stress-energy tensor:

$$T^{ab} = (\rho + P/c^2)u^a u^b + P\eta^{ab},$$

where ρ is the proper mass density in the fluid rest frame and P is the pressure. Show that in the Newtonian limit these equations reduce to the equation of continuity and the Euler equation for fluid motion.

The stress-energy tensor for a perfect fluid is given by

$$T^{ab} = \left(\rho + \frac{P}{c^2}\right)u^a u^b + P\eta^{ab} \tag{1}$$

in flat spacetime.

We want to write out

$$\nabla_a T^{ab} = 0 \tag{2}$$

for the components.

In flat space,

$$\eta^{ab} = \mathrm{diag}(-1,1,1,1) \tag{3}$$

We have the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4}$$

We note that

$$u^a = \gamma(c, v^i). \tag{5}$$

Let's start with writing out the components.
This isn't as simple in a non-comoving frame.

For T^{00},

$$T^{00} = \left(\rho + \frac{P}{c^2}\right)u^0u^0 + P\eta^{00} \qquad (6)$$

$$T^{00} = \left(\rho + \frac{P}{c^2}\right)(\gamma c)(\gamma c) + P(-1) \qquad (7)$$

$$T^{00} = \left(\rho + \frac{P}{c^2}\right)\gamma^2 c^2 - P. \qquad (8)$$

For T^{0i},

$$T^{0i} = \left(\rho + \frac{P}{c^2}\right)u^0u^i + P\eta^{0i} \qquad (9)$$

$$T^{0i} = \left(\rho + \frac{P}{c^2}\right)(\gamma c)(\gamma v^i) + P(0) \qquad (10)$$

$$T^{0i} = \left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i c \qquad (11)$$

It is easy to see that this also describes T^{i0}

$$T^{i0} = \left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i c. \qquad (12)$$

For T^{ij},

$$T^{ij} = \left(\rho + \frac{P}{c^2}\right)u^i u^j + P h^{ij} \tag{13}$$

$$T^{ij} = \left(\rho + \frac{P}{c^2}\right)(\gamma v^i)(\gamma v^j) + P\delta^{ij} \tag{14}$$

$$T^{ij} = \left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i v^j + P\delta^{ij}. \tag{15}$$

We can now deal with conservation using (2).

Since we're dealing in what is essentially an SR setting, the Principle of minimal coupling allows (2) to become

$$\nabla_a T^{ab} \rightarrow \partial_a T^{ab}. \tag{16}$$

It is easier to work with

$$\partial_a T^{ab} \rightarrow \partial_b T^{ab},$$

Since T^{ab} is symmetric

Note that

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}. \tag{17}$$

For $a = 0$, we have

$$\partial_b T^{0b} = 0, \tag{18}$$

$$\partial_0 T^{00} + \partial_i T^{0i} = 0, \tag{19}$$

which leads to

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 c^2 - P\right) + \frac{\partial}{\partial x^i}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i c\right) = 0. \tag{20}$$

For $a = i$, we have

$$\partial_b T^{ib} = 0 \tag{21}$$

$$\partial_0 T^{i0} + \partial_j T^{ij} = 0, \tag{22}$$

which leads to

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i c\right) + \frac{\partial}{\partial x^j}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i v^j + P\delta^{ij}\right) = 0. \tag{23}$$

We now seek the Newtonian limit

$$c \to +\infty. \tag{24}$$

Note that

$$\lim_{c \to +\infty} \gamma = 1. \tag{25}$$

Dividing (20) by c yields

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 c^2 - P\right) + \frac{\partial}{\partial x^i}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i\right) = 0, \tag{26}$$

$$\frac{\partial}{\partial t}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 - \frac{P}{c^2}\right) + \frac{\partial}{\partial x^i}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 v^i\right) = 0. \tag{27}$$

Taking the Newtonian limit, considering P is constant wrt. t, gives

$$\lim_{c \to +\infty}\left(\frac{\partial}{\partial t}\left(\left(\rho + \frac{P}{c^2}\right)\gamma^2 - \frac{P}{c^2}\right)\right) = \frac{\partial \rho}{\partial t} \tag{28}$$

and

$$\lim_{c \to +\infty} \left(\frac{\partial}{\partial x^i} \left(\left(\rho + \frac{P}{c^2} \right) \gamma^2 v^i \right) \right) = \frac{\partial}{\partial x^i} \left(\rho v^i \right) \quad . \qquad (29)$$

Adding these gives

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i} \left(\rho v^i \right) = 0} \quad . \qquad (30)$$

This is indeed equivalent to the continuity equation in terms of the Eulerian time derivative

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{v} \right) \quad . \qquad (31)$$

From (23), we take the Newtonian limit, which gives

$$\lim_{c \to +\infty} \left(\frac{1}{c} \frac{\partial}{\partial t} \left(\left(\rho + \frac{P}{c^2} \right) \gamma^2 v^i c \right) \right) = \frac{\partial}{\partial t} \left(\rho v^i \right) \qquad (32)$$

$$\lim_{c \to +\infty} \left(\frac{\partial}{\partial x^j} \left(\left(\rho + \frac{P}{c^2} \right) \gamma^2 v^i v^j + P \delta^{ij} \right) \right) = \frac{\partial}{\partial x^j} \left(\rho v^i v^j + P \delta^{ij} \right) \quad . \qquad (33)$$

Adding these, we find

$$\frac{\partial}{\partial t} \left(\rho v^i \right) + \frac{\partial}{\partial x^j} \left(\rho v^i v^j + P \delta^{ij} \right) = 0 \qquad (34)$$

$$v^i \frac{\partial \rho}{\partial t} + \rho \frac{\partial v^i}{\partial t} + \frac{\partial}{\partial x^j} \left(\rho v^i v^j + P \delta^{ij} \right) = 0 \quad . \qquad (35)$$

Expanding

$$\frac{\partial}{\partial x^j}\left(\rho v^i v^j\right) = v^i \frac{\partial}{\partial x^j}(\rho v^j) + \rho\left(v^j \frac{\partial}{\partial x^j}\right)(v^i) . \qquad (36)$$

From (30),

$$v^i \frac{\partial \rho}{\partial t} = - v^i \frac{\partial}{\partial x^j}(\rho v^j) . \qquad (37)$$

(35) then simplifies to

$$\rho \frac{\partial v^i}{\partial t} + \rho\left(v^j \frac{\partial}{\partial x^j}\right)(v^i) + \frac{\partial}{\partial x^j}\left(P \delta^{ij}\right) = 0 \qquad (38)$$

Dividing by ρ gives

$$\boxed{\frac{\partial v^i}{\partial t} + \left(v^j \frac{\partial}{\partial x^j}\right)(v^i) + \frac{1}{\rho}\frac{\partial}{\partial x^j}\left(P \delta^{ij}\right) = 0} . \qquad (39)$$

This is indeed equivalent to the Euler Equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla} P = 0 . \qquad (40)$$